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EXOTIC SMOOTH STRUCTURES ON SPHERES

Autor: Alejandro García Rivas

Director: Dr. Ricardo García López
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Abstract

The main goal of this work is to prove the existence of exotic smooth spheres. These are smooth manifolds that are homeomorphic but not diffeomorphic to the standard sphere. This was first shown in the 7-dimensional case by John Milnor in his influential paper [Mil56], and this work replicates his construction.

In order to state and prove this result, though, a journey through some background is needed. This includes singular homology and cohomology theory, Morse theory and characteristic classes. Hence, we also slightly develop these topics here.

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Contents

Introduction

As John Milnor explains in [Mil07], back in the 1950's he was studying $2n$-manifolds that are $(n-1)$-connected. His research allowed him to make the following statement:

There exist homotopy 7-spheres that are not diffeomorphic to the standard 7-sphere.

This led him to believe he had found a counterexample to the Poincaré conjecture in dimension 7, as he assumed that two homeomorphic manifolds are always diffeomorphic. However, using Morse theory, he managed to show that these homotopy 7-spheres were actually topological 7-spheres, thus proving the existence of exotic spheres in dimension 7. This is considered to be a remarkable landmark in the history of differential geometry as it solved an interesting problem and, at the same time, it created a wide scope of new lines of research.

In this work we start by introducing basic results of singular homology and cohomology in Chapter 1. These are presented without proofs, but several references are given.

In Chapter 2, we give a small introduction to Morse Theory. In particular, we state and prove Reeb's theorem and use it to show that certain smooth 7-manifolds M_k^7 are homeomorphic to the 7-sphere. These manifolds will be seen later on to be exotic 7-spheres for suitable values of k.

In Chapter 3, we define three types of characteristic classes. Namely, the Euler class, the Chern classes and the Pontrjagin classes. We also state their important properties and give proofs for some of them. Finally, we partially prove the Hirzebruch signature theorem, as it constitutes a fundamental building block for Milnor's construction of exotic 7-spheres.

Having introduced all these topics, in Chapter 4 we use them to finally prove that the 7-manifolds M_k^7 are not diffeomorphic to the standard 7-sphere when $k^2 \not\equiv 1 \pmod 7$, thus completing the goal of this work.

Chapter 1

Preliminary topics

This chapter is a review of important definitions and results about homology and cohomology theory. The content is mostly taken from [Hat01], but some results, arguments and notation also come from [GH81] and [May99].

1.1 Homology

Throughout this section, let X, Y be non-empty topological spaces and let R be a commutative ring with unit. Also, every subset of $\mathbf{R}^m$ is assumed to have the induced topology.

We begin with a purely algebraic definition.

Definition 1.1. A *chain complex* $(C_\bullet, \partial_\bullet)$ is a sequence of R-modules C_n and R-linear maps $\partial_n : C_n \to C_{n-1}$ that satisfy $\partial_n \circ \partial_{n+1} = 0$. We will sometimes drop the index n of the maps ∂_n, and hence we may write

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \longrightarrow \cdots$$

to denote the chain complex $(C_\bullet, \partial_\bullet)$. The n^{th} *homology R-module of* $(C_\bullet, \partial_\bullet)$ is defined to be the quotient $H_n(C) = H_n(C_\bullet, \partial_\bullet) := \ker \partial_n / \operatorname{im} \partial_{n+1}$.

A *chain map* $f_\bullet : (C_\bullet, \partial_\bullet) \to (D_\bullet, \partial_\bullet)$ between chain complexes is a collection of R-linear maps $f_n : C_n \to D_n$ such that the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial"] & D_n \arrow[r, "\partial"] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

commutes. As a consequence, $f_n(\ker \partial_n) \subseteq \ker \partial_n$ and $f_n(\operatorname{im} \partial_{n+1}) \subseteq \operatorname{im} \partial_{n+1}$ for every integer n, so $f_\bullet$ induces R-linear maps in homology

$$\begin{aligned} H_n(f_\bullet) : H_n(C) &\to H_n(D) \\ [c] &\mapsto [f_n(c)]. \end{aligned}$$

Our first goal is to construct a chain complex $(C_\bullet(X), \partial_\bullet)$ for every topological space X, so that the homology of X can be taken to be the homology of $(C_\bullet(X), \partial_\bullet)$. There are several ways to do this if X is furnished with additional structure. For instance, one may consider simplicial homology for simplicial complexes or cellular homology for CW complexes. In order to be as general as possible, we will only consider singular homology, as no additional structure on X is required.

Definition 1.2. Let $v_0, \dots, v_n \in \mathbf{R}^{n+1}$ be $n+1$ linearly independent points. The *n-simplex* spanned by $v_0, \dots, v_n$ (denoted $[v_0, \dots, v_n]$) is the smallest convex subset of $\mathbf{R}^{n+1}$ containing them, namely:

$$[v_0, \dots, v_n] = \left\{ \sum_{i=0}^{n} t_i v_i \in \mathbf{R}^{n+1} : \sum_{i=0}^{n} t_i = 1, t_i \geq 0 \right\} \subseteq \mathbf{R}^{n+1}.$$

The *standard n-simplex* (denoted Δ^n) is the n-simplex $[e_0, \dots, e_n]$, where $e_i = (0, \dots, \overbrace{1}^{i+1}, \dots, 0)$. Namely:

$$\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbf{R}^{n+1} : \sum_{i=0}^{n} t_i = 1, t_i \geq 0 \right\} \subseteq \mathbf{R}^{n+1}.$$

Definition 1.3. A *singular n-simplex* is a continuous map $\Delta^n \to X$. We denote by $C_n(X)$ the free R-module generated by all singular n-simplexes and call its elements *(singular) n-chains*. For an n-simplex $\sigma : \Delta^n \to X$, let $\sigma^{(i)} : \Delta^{n-1} \to X$ be the composition map

$$\begin{tikzcd} & {[e_0, \dots, \hat{e}_i, \dots, e_n]} \arrow[dr, "\sigma_{|[e_0,\dots,\hat{e}_i,\dots,e_n]}"] & \\ \Delta^{n-1} \arrow[ur, "\phi^{(i)}_{|\Delta^{n-1}}"] & & X \end{tikzcd}$$

where $\phi^{(i)} : \mathbf{R}^n \to \mathbf{R}^{n+1}$ is the unique affine map satisfying for $0 \leq j \leq n-1$

$$\phi^{(i)}(e_j) = \begin{cases} e_j & \text{if } j < i, \\ e_{j+1} & \text{if } j \geq i. \end{cases}$$

We also define the *boundary map* $\partial_n : C_n(X) \to C_{n-1}(X)$ by

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma^{(i)}$$

for a singular n-simplex $\sigma : \Delta^n \to X$, and we extend linearly to any n-chain.

Lemma 1.4. $\partial_n \partial_{n+1} = 0$ *for every* $n \geq 0$.

As a consequence, we have a chain complex of R-modules

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

that allows us to define the homology of an arbitrary topological space X.

Definition 1.5. Elements of $Z_n(X) := \ker \partial_n$ are called *n-cycles* and elements of $B_n(X) := \operatorname{im} \partial_{n+1}$ are called *n-boundaries*. The *n^{th} (singular) homology R-module of X* is defined as $H_n(X) := Z_n(X)/B_n(X)$. If we want to make explicit the ring R we are working with, we will write $H_n(X; R)$, $C_n(X; R)$ and so on.

Remark 1.6. The previous chain complex can be slightly augmented to

$$\cdots \longrightarrow C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} R \longrightarrow 0$$

where $\epsilon(\sigma) = 1$ for every 0-simplex σ. The homology R-modules of this extended chain complex, $\widetilde{H}_n(X)$ (or $\widetilde{H}_n(X; R)$), are called the *reduced homology R-modules* and they satisfy

$$H_n(X) = \begin{cases} \widetilde{H}_0(X) \oplus R, & \text{if } n = 0, \\ \widetilde{H}_n(X), & \text{if } n > 0. \end{cases}$$

Proposition 1.7. *Let $X = \bigcup_{i \in I} X_i$ be the decomposition of X into its path-connected components X_i, $i \in I$.*

(i) There is an isomorphism $H_n(X) \cong \bigoplus_{i \in I} H_n(X_i)$.

(ii) $H_0(X) \cong \bigoplus_{i \in I} R$.

(iii) If X is path-connected, then $H_1(X; \mathbf{Z}) \cong \pi_1(X)^{Ab}$.

Definition 1.8. A continuous map $f : X \to Y$ induces a family of homomorphisms

$$C_n(f) : C_n(X) \to C_n(Y)$$

by precomposition. Namely, $C_n(f)$ maps $\sigma : \Delta^n \to X$ to $f\sigma : \Delta^n \to X \to Y$ and as usual this is extended linearly to n-chains.

This family of morphisms makes the diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "C_{n+1}(f)"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "C_n(f)"] & C_{n-1}(X) \arrow[r, "\partial"] \arrow[d, "C_{n-1}(f)"] & \cdots \\
\cdots \arrow[r, "\partial"] & C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

commute. Hence $C_\bullet(f)$ is a chain map and it induces morphisms in homology $H_n(f) : H_n(X) \to H_n(Y)$. We may also write f_* to denote $C_n(f)$ or $H_n(f)$ for any value of n as long as its meaning is clear from the context.

Remark 1.9. H_n is a covariant functor between the category of topological spaces and the category of R-modules, i.e. $H_n(\mathrm{id}) = \mathrm{id}$ and $H_n(fg) = H_n(f)H_n(g)$ for $g : X \to Y, f : Y \to Z$.

Theorem 1.10. *If two maps $f, g : X \to Y$ are homotopic, then they induce the same homomorphisms in homology, i.e. $H_n(f) = H_n(g)$.*

Corollary 1.11. *If $f : X \to Y$ is a homotopy equivalence, then $H_n(f) : H_n(X) \to H_n(Y)$ is an isomorphism.*

Let us now generalize the definition of homology to that of *relative homology*.

Definition 1.12. Let $A \subseteq X$ (with the induced topology) and let $C_n(X, A) := C_n(X)/C_n(A)$ (where $C_n(A)$ is identified with $C_n(i)(C_n(A))$, $i : A \hookrightarrow X$ the inclusion map). Since $\partial : C_n(X) \to C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$, we have well-defined morphisms $\bar{\partial} : C_n(X, A) \to C_{n-1}(X, A)$ satisfying $\bar{\partial}^2 = 0$. Thus, we have a chain complex

$$\cdots \xrightarrow{\bar{\partial}_{n+1}} C_n(X, A) \xrightarrow{\bar{\partial}_n} C_{n-1}(X, A) \xrightarrow{\bar{\partial}_{n-1}} \cdots \xrightarrow{\bar{\partial}_1} C_0(X, A) \xrightarrow{\bar{\partial}_0} 0$$

that allows us to define the n^{th} *(singular) relative homology R-module of* (X, A) as $H_n(X, A) := \ker \bar{\partial}_n / \operatorname{im} \bar{\partial}_{n+1}$. Again, if we want to make explicit the ring R we are working with, we will write $H_n(X, A; R)$, $C_n(X, A; R)$.

Remark 1.13. Let $Z_n(X, A)$ be the submodule of $C_n(X)$ consisting of the n-chains $\alpha \in C_n(X)$ such that $\partial\alpha \in C_{n-1}(A)$. We call its elements *relative n-cycles*.

Let $B_n(X, A)$ be the submodule of $C_n(X)$ consisting of the n-chains $\alpha \in C_n(X)$ for which there is an n-chain $\beta \in C_n(A)$ such that $\alpha - \beta$ is an n-boundary on X. We call its elements *relative n-boundaries*.

The first isomorphism theorem implies $H_n(X, A) \cong Z_n(X, A)/B_n(X, A)$, which is sometimes a nicer description of the relative homology modules.

Definition 1.14. A continuous map $f : (X, A) \to (Y, B)$ (i.e. $f : X \to Y$ and $f(A) \subseteq B$) induces a family of chain maps $C_n(f) : C_n(X, A) \to C_n(Y, B)$ by precomposition and this induces homomorphisms in homology $H_n(f) : H_n(X, A) \to H_n(Y, B)$. As before, we may also write f_* to denote any of these homomorphisms.

Theorem 1.15. *If two maps $f, g : (X, A) \to (Y, B)$ are homotopic through maps of pairs $(X, A) \to (Y, B)$, then they induce the same homomorphism, i.e. $H_n(f) = H_n(g) : H_n(X, A) \to H_n(Y, B)$.*

Lemma 1.16. *A short exact sequence of chain complexes*

$$0 \longrightarrow A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{j} C_\bullet \longrightarrow 0,$$

where i and j are chain maps, naturally induces a long exact sequence in homology

$$\cdots \longrightarrow H_n(A) \xrightarrow{H_n(i)} H_n(B) \xrightarrow{H_n(j)} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{H_{n-1}(i)} H_{n-1}(B) \longrightarrow \cdots$$

where $\partial : H_n(C) \to H_{n-1}(A)$ is the map given by the following diagram chasing: take $[c] \in H_n(C)$ with $c \in Z_n(C)$; choose $b \in B_n$ such that $j(b) = c$; choose $a \in A_{n-1}$ such that $i(a) = \partial b$; define $\partial[c] = [a]$.

Remark 1.17. Here the word *naturally* refers to the fact that if

$$\begin{tikzcd} 0 \arrow[r] & A_\bullet \arrow[r, "i"] \arrow[d, "\alpha"] & B_\bullet \arrow[r, "j"] \arrow[d, "\beta"] & C_\bullet \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A'_\bullet \arrow[r, "i'"] & B'_\bullet \arrow[r, "j'"] & C'_\bullet \arrow[r] & 0 \end{tikzcd}$$

commutes, then the induced diagram

$$\begin{tikzcd} \cdots \arrow[r] & H_n(A) \arrow[r, "H_n(i)"] \arrow[d, "H_n(\alpha)"] & H_n(B) \arrow[r, "H_n(j)"] \arrow[d, "H_n(\beta)"] & H_n(C) \arrow[r, "\partial"] \arrow[d, "H_n(\gamma)"] & H_{n-1}(A) \arrow[r] \arrow[d, "H_{n-1}(\alpha)"] & \cdots \\ \cdots \arrow[r] & H_n(A') \arrow[r, "H_n(i')"] & H_n(B') \arrow[r, "H_n(j')"] & H_n(C') \arrow[r, "\partial"] & H_{n-1}(A') \arrow[r] & \cdots \end{tikzcd}$$

also commutes. This property is commonly known as *naturality* or *functoriality* in the literature.

Corollary 1.18. *For $A \subseteq X$, the short exact sequence*

$$0 \longrightarrow C_\bullet(A) \xrightarrow{i} C_\bullet(X) \xrightarrow{j} C_\bullet(X, A) \longrightarrow 0$$

(where i is the inclusion and j is the projection) naturally induces a long exact sequence in homology

$$\cdots \to H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(j)} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots \to H_0(X, A) \to 0.$$

Remark 1.19. Using augmented chain complexes, the previous long exact sequence can be terminated with

$$\cdots \longrightarrow H_1(X, A) \longrightarrow \widetilde{H}_0(A) \longrightarrow \widetilde{H}_0(X) \longrightarrow H_0(X, A) \longrightarrow 0.$$

Theorem 1.20. *(Universal coefficient theorem for homology) Let $(C_\bullet, \partial_\bullet)$ be a chain complex of free $\mathbf{Z}$-modules and let R be ring. Then, there are natural short exact sequences*

$$0 \longrightarrow H_n(C) \otimes R \longrightarrow H_n(C; R) \longrightarrow \mathrm{Tor}(H_{n-1}(C), R) \longrightarrow 0$$

for every integer n and all these sequences split.

Theorem 1.21. *(Excision) Let $Z \subseteq A \subseteq X$ such that the closure of Z is contained in the interior of A. Then the inclusion $i : (X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces isomorphisms in homology*

$$H_n(i) : H_n(X \setminus Z, A \setminus Z) \to H_n(X, A).$$

Theorem 1.22. *(Mayer-Vietoris) Let $A, B \subseteq X$ such that $A \cap B \neq \emptyset$ and X is the union of the interiors of A and B. Then, there is a natural long exact sequence called the Mayer-Vietoris sequence*

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A \cap B) \arrow[r, "\Phi"] & H_n(A) \oplus H_n(B) \arrow[r, "\Psi"] & H_n(X) \arrow[dll, "\partial"'] & \\
& H_{n-1}(A \cap B) \arrow[r, "\Phi"] & H_{n-1}(A) \oplus H_{n-1}(B) \arrow[r, "\Psi"] & H_{n-1}(X) \arrow[r] & \cdots \arrow[dllll] \\
H_1(X) \arrow[r] & \widetilde{H}_0(A \cap B) \arrow[r] & \widetilde{H}_0(A) \oplus \widetilde{H}_0(B) \arrow[r] & \widetilde{H}_0(X) \arrow[r] & 0
\end{tikzcd}$$

where

(i) $i_A : A \cap B \hookrightarrow A$ and $i_B : A \cap B \hookrightarrow B$ are the inclusions and $\Phi(\gamma) = (H_n(i_A)(\gamma), -H_n(i_B)(\gamma))$ for $\gamma \in H_n(A \cap B)$;

(ii) $j_A : A \hookrightarrow X$ and $j_B : B \hookrightarrow X$ are the inclusions and $\Psi(\alpha, \beta) = H_n(j_A)(\alpha) + H_n(j_B)(\beta)$ for $\alpha \in H_n(A)$ and $\beta \in H_n(B)$;

(iii) Any $\gamma \in H_n(X)$ can be represented by a cycle of the form $a + b$ with $a \in C_n(A)$ and $b \in C_n(B)$. Then take $\partial\gamma = [\partial a] = [-\partial b]$.

Remark 1.23. One can analogously define relative Mayer-Vietoris sequences. Let $(X, Y) = (A \cup B, C \cup D)$ with $Y \subseteq X, C \subseteq A, D \subseteq B$ such that X is the union of the interiors of A and B and Y is the union of the interiors of C and D. Then, the relative Mayer-Vietoris sequence is given by

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A \cap B, C \cap D) \arrow[r, "\Phi"] & H_n(A, C) \oplus H_n(B, D) \arrow[r, "\Psi"] & H_n(X, Y) \arrow[dll, "\partial"'] & \\
& H_{n-1}(A \cap B, C \cap D) \arrow[r] & \cdots \arrow[r] & H_0(X, Y) \arrow[r] & 0.
\end{tikzcd}$$

As an application of these results, let us compute the homology of a point and the homology of the spheres.

Example 1.24. (Homology of a point) Let $X = \{p\}$. Since there is only one map $\Delta^n \to X$, we have $C_n(X) \cong R$ for every $n \geq 0$. The boundary map ∂_n will then be the identity or the zero map depending on the parity of n. Namely:

$$\partial_n = \begin{cases} 0, & n = 0 \text{ or } n \text{ odd.} \\ \text{id}, & \text{otherwise.} \end{cases}$$

Thus, we have the following chain complex

$$\cdots \xrightarrow{id} R \xrightarrow{0} R \xrightarrow{id} R \xrightarrow{0} R \xrightarrow{0} 0$$

and we finally get

$$H_n(\{p\}) = \begin{cases} R, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

or simply $\widetilde{H}_n(\{p\}) = 0$ for every $n \geq 0$.

Example 1.25. (Homology of the spheres) By the previous example and Proposition 1.7 we already got

$$H_n(\mathbb{S}^0) = \begin{cases} R \oplus R, & n = 0. \\ 0, & n > 0. \end{cases}$$

$$H_0(\mathbb{S}^m) = \begin{cases} R \oplus R, & m = 0. \\ R, & m > 0. \end{cases}$$

We now set $X = \mathbb{S}^m (m > 0)$, $A = \mathbb{S}^m \setminus \{\text{north pole}\}$, $B = \mathbb{S}^m \setminus \{\text{south pole}\}$. Notice that A and B can be deformation retracted to a point, so $H_n(A) = H_n(B) = 0$ for $n > 0$ and $H_0(A) = H_0(B) = R$. Also $A \cap B$ can be deformation retracted to the equator, i.e. to $\mathbb{S}^{m-1}$. We then have a Mayer-Vietoris sequence

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & H_n(\mathbb{S}^m) \arrow[r] & H_{n-1}(\mathbb{S}^{m-1}) \arrow[r] & 0 \arrow[dllll] \\
\cdots \arrow[r] & 0 \arrow[r] & H_2(\mathbb{S}^m) \arrow[r] & H_1(\mathbb{S}^{m-1}) \arrow[r] & 0 \arrow[dll] \\
& & H_1(\mathbb{S}^m) \arrow[r] & \widetilde{H}_0(\mathbb{S}^{m-1}) \arrow[r] & 0 \arrow[dll] \\
& & 0 \arrow[r] & 0. &
\end{tikzcd}$$

In particular, $\widetilde{H}_n(\mathbb{S}^m) \cong \widetilde{H}_{n-1}(\mathbb{S}^{m-1})$ for $n \geq 1, m \geq 1$. By induction and the previous facts, we have

$$H_n(\mathbb{S}^m) = \begin{cases} R, & \text{if } n = m > 0, \\ 0, & \text{if } n \neq m. \end{cases}$$

1.2 Orientation of Manifolds

Unless otherwise stated, let M be an m-dimensional topological manifold. Given a subspace $A \subseteq M$ we will use the notation $H_n(M|A; R) = H_n(M, M \setminus A; R)$. Whenever A is a single point $\{x\}$, we will write $H_n(M|x; R) = H_n(M|\{x\}; R)$.

Remark 1.26. Notice that for $x \in M$ we have a chain of isomorphisms

$$H_i(M|x; R) \cong H_i(\mathbf{R}^m, \mathbf{R}^m \setminus \{0\}; R) \cong \widetilde{H}_{i-1}(\mathbf{R}^m \setminus \{0\}; R) \cong \widetilde{H}_{i-1}(\mathbb{S}^{m-1}; R) \cong \begin{cases} R, & \text{if } i = m, \\ 0, & \text{if } i \neq m. \end{cases}$$

Analogously, for a ball of finite radius $B \subseteq M$, we have

$$H_i(M|B; R) \cong \begin{cases} R, & \text{if } i = m, \\ 0, & \text{if } i \neq m. \end{cases}$$

This leads to the following definition.

Definition 1.27. A local R-orientation of M at a point $x \in M$ is a choice of a generator (i.e. invertible element) $\mu_x \in H_m(M|x; R) \cong R$.

Let $M_R = \{\mu_x : x \in M, \mu_x \in H_n(M|x; R)\}$. We wish to equip M_R with a topology by specifying a basis of open sets. For every open ball of finite radius $B \subseteq M$ and an element $\mu_B \in H_m(M|B; R)$, let $U(\mu_B) = \{\mu_x \in M_R : x \in B, j_B^x(\mu_B) = \mu_x\}$ where $j_B^x : H_m(M|B; R) \to H_m(M|x; R)$ is the map induced by inclusion. The family $\{U(\mu_B)\}_B$ for varying B and μ_B is the basis of a topology on M_R. One can check that this topological space is locally homeomorphic to $\mathbf{R}^m$, Hausdorff and second-countable (as long as R is countable), so M_R is a topological m-manifold. Also, the projection $p : M_R \to M$, $p(\mu_x) = x$ is a covering space.

Definition 1.28. A (global) R-orientation of M is a continuous section of $p : M_R \to M$ that assigns to each $x \in M$ a generator $\mu_x \in H_m(M|x; R)$. If M admits an R-orientation, we say that M is R-orientable. If, in addition, an R-orientation of M has been fixed, we say that M is R-oriented.

Theorem 1.29. *Let M be a closed connected m-manifold. Then:*

(i) If M is R-orientable, the map $j^x : H_m(M; R) \to H_m(M|x; R) \cong R$ induced by the inclusion $(M, \emptyset) \hookrightarrow (M, M \setminus \{x\})$ is an isomorphism for all $x \in M$.

(ii) If M is not R-orientable, the map $j^x : H_m(M; R) \to H_m(M|x; R) \cong R$ induced by the inclusion $(M, \emptyset) \hookrightarrow (M, M \setminus \{x\})$ is injective with image $\{r \in R : 2r = 0\}$ for all $x \in M$.

(iii) $H_i(M; R) = 0$ for $i > m$.

Definition 1.30. An element of $H_m(M; R)$ whose image by j^x is a generator of $H_m(M|x; R)$ for every $x \in M$ is called a *fundamental class* of M with coefficients in R. By the theorem, a connected manifold M has a fundamental class with coefficients in R if, and only if, it is closed and R-orientable.

Remark 1.31. If $R = \mathbf{Z}$, we will omit any reference to the ring in the concepts defined above and below. For example, we will simply say orientation, orientable, oriented instead of $\mathbf{Z}$-orientation, $\mathbf{Z}$-orientable, $\mathbf{Z}$-oriented, respectively.

One can also orientate manifolds with boundary using homology classes. We restrict ourselves to the differentiable and compact case, as this is the only case we will need for this work. From now on until the end of this section, let M be a smooth, compact m-manifold with boundary.

Definition 1.32. An R-orientation of M is an R-orientation of its interior $\mathring{M} = M \setminus \partial M$.

The following result is of great importance for this section and next ones.

Theorem 1.33. *(Smooth collar neighborhood) There is an open neighborhood of ∂M in M which is diffeomorphic to $\partial M \times [0, 1)$ under a map that identifies ∂M with $\partial M \times \{0\}$. Such a neighborhood is called a collar neighborhood of M.*

As a consequence, the inclusion $\mathring{M} \hookrightarrow M$ is a homotopy equivalence.

Proposition 1.34. *An R-orientation of M determines an R-orientation of ∂M.*

Proofs of these results and of the one below can be found in chapter 21 of [May99].

Proposition 1.35. *If M is R-oriented, then the connecting homomorphism*

$$\partial : H_n(M, \partial M; R) \to H_{n-1}(\partial M; R)$$

of the long exact sequence of the pair $(M, \partial M)$ is an isomorphism.

These last propositions allow us to make the following definition.

Definition 1.36. If M is R-oriented and $[\partial M]$ is the fundamental class for ∂M with coefficients in R, we define the *fundamental class* of M with coefficients in R to be the unique element $[M, \partial M] \in H_n(M, \partial M; R)$ that satisfies $\partial[M, \partial M] = [\partial M]$.

1.3 Cohomology

Throughout this section let X, Y be non-empty topological spaces and let R be a commutative ring with unit. The first definitions and results below are purely algebraic with no topology involved.

Definition 1.37. A *cochain complex* $(C^\bullet, \delta^\bullet)$ is a sequence of R-modules C^n and R-linear maps $\delta^n : C^{n-1} \to C^n$ that satisfy $\delta^{n+1} \circ \delta^n = 0$. As with chain complexes, we will sometimes drop the index n of the maps δ^n and write

$$\cdots \longrightarrow C^{n-1} \xrightarrow{\delta} C^n \xrightarrow{\delta} C^{n+1} \longrightarrow \cdots$$

to denote the complex $(C^\bullet, \delta^\bullet)$. The n^{th} *cohomology R-module of* $(C^\bullet, \delta^\bullet)$ is defined to be the quotient $H^n(C) = H^n(C^\bullet, \delta^\bullet) := \ker \delta^{n+1} / \operatorname{im} \delta^n$.

Analogously to chain complexes, a *cochain map* $f^\bullet : (C^\bullet, \delta^\bullet) \to (D^\bullet, \delta^\bullet)$ is a collection of R-linear maps $f^n : C^n \to D^n$ such that the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C^{n-1} \arrow[r, "\delta"] \arrow[d, "f^{n-1}"] & C^n \arrow[r, "\delta"] \arrow[d, "f^n"] & C^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \cdots \\
\cdots \arrow[r] & D^{n-1} \arrow[r, "\delta"] & D^n \arrow[r, "\delta"] & D^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

commutes. As a consequence, $f^\bullet$ induces well-defined R-linear maps in cohomology

$$\begin{aligned} H^n(f^\bullet) : H^n(C) &\to H^n(D) \\ [c] &\mapsto [f^n(c)]. \end{aligned}$$

Given an R-module A, recall that its dual is the R-module $A^* := \operatorname{Hom}_R(A, R)$. Furthermore, given an R-linear map $\alpha : A \to B$ between R-modules, the dual map of α is defined as

$$\begin{aligned} \alpha^* : B^* &\to A^* \\ \varphi &\mapsto \varphi \circ \alpha. \end{aligned}$$

Remark 1.38. It is worth noting that we can associate a cochain complex $(C^\bullet, \delta^\bullet)$ to any chain complex $(C_\bullet, \partial_\bullet)$. Indeed, we may take C^n to be C_n^*, and δ^n to be ∂_n^*. A chain map $f_\bullet : (C_\bullet, \partial_\bullet) \to (D_\bullet, \partial_\bullet)$ may also be associated to a cochain map $f^\bullet : (D^\bullet, \delta^\bullet) \to (C^\bullet, \delta^\bullet)$ by taking f^n to be f_n^*. This construction allows us to define the n^{th} *cohomology R-module of* $(C_\bullet, \partial_\bullet)$ as the n^{th} cohomology R-module of $(C^\bullet, \delta^\bullet)$, denoted by $H^n(C_\bullet, \partial_\bullet)$, or simply $H^n(C)$.

Homology and cohomology modules of free chain complexes are related by the following result.

Theorem 1.39. *(Universal coefficient theorem for cohomology) Assume that R is a principal ideal domain. Let $(C_\bullet, \partial_\bullet)$ be a chain complex where every C_n is a free R-module. Then, for every integer n there is a split short exact sequence*

$$0 \longrightarrow \operatorname{Ext}_R(H_{n-1}(C), R) \longrightarrow H^n(C; R) \xrightarrow{h} \operatorname{Hom}_R(H_n(C), R) \longrightarrow 0.$$

Here, the map h is defined by taking every $[\varphi] \in H^n(C; R)$ to the assignment

$$\begin{aligned} h([\varphi]) : H_n(C) &\to R \\ [\alpha] &\mapsto \langle [\varphi], [\alpha] \rangle := \varphi(\alpha) \end{aligned}$$

Remark 1.40. The fact that these sequences split implies that

$$H^n(C; R) \cong \operatorname{Ext}_R(H_{n-1}(C), R) \oplus \operatorname{Hom}_R(H_n(C), R),$$

so the cohomology modules of a chain complex of free modules are determined by its homology modules, although not naturally.

Remark 1.38 also enables us to make the jump to cohomology of topological spaces.

Definition 1.41. The n^{th} *cohomology R-module of X* is simply the n^{th} cohomology R-module of the chain complex $(C_\bullet(X;R), \partial_\bullet)$.

For clarity, we now unravel this definition. We write $C^n(X;R) = C_n(X;R)^*$ and call its elements *n-cochains*. We call $\delta^n = \partial_n^* : C^{n-1}(X;R) \to C^n(X;R)$ the *coboundary map*. It explicitly acts on a cochain $\varphi \in C^n(X;R)$ by

$$\delta\varphi(\sigma) = \sum_{i=0}^{n+1} (-1)^i \varphi(\sigma^{(i)}), \quad \text{for any simplex } \sigma : \Delta^{n+1} \to X.$$

Let $Z^n(X;R) = \ker \delta^{n+1}$ and call its elements *n-cocycles*. Let $B^n(X;R) = \operatorname{im} \delta^n$ and call its elements *n-coboundaries*. Finally, the n^{th} cohomology R-module of X is then $H^n(X;R) = Z^n(X;R)/B^n(X;R)$.

Definition 1.42. Let $A \subseteq X$. The n^{th} *relative cohomology R-module of the pair* (X, A), denoted by $H^n(X, A; R)$, is the n^{th} cohomology R-module of the chain complex $(C_\bullet(X, A; R), \bar{\partial}_\bullet)$ of Definition 1.12. This definition may be unraveled in a similar manner as the one above.

The following facts are just translations of the results mentioned in Section 1.1 into cohomology.

Definition 1.43. A continuous map $f : (X, A) \to (Y, B)$ induces a chain map $C_\bullet(f)$ (cf. Definition 1.14). By taking the duals $C^n(f) := C_n(f)^* : C^n(Y, B; R) \to C^n(X, A; R)$, we obtain a cochain map $C^\bullet(f)$ that induces R-linear maps in cohomology $H^n(f) : H^n(Y, B; R) \to H^n(X, A; R)$. We may also denote $C^n(f)$ and $H^n(f)$ by f^*, as long as the meaning is clear from the context.

Remark 1.44. H^n is a contracovariant functor between the category of topological spaces and the category of R-modules, i.e. $H^n(\mathrm{id}) = \mathrm{id}$ and $H^n(fg) = H^n(g)H^n(f)$ for $g : X \to Y, f : Y \to Z$.

Theorem 1.45. *If two maps $f, g : (X, A) \to (Y, B)$ are homotopic through maps from (X, A) to (Y, B), then they induce the same homomorphisms in cohomology, i.e. $H^n(f) = H^n(g)$.*

Corollary 1.46. *Dualizing the short exact sequence of Corollary 1.18, we obtain a short exact sequence*

$$0 \longleftarrow C^n(A;R) \xleftarrow{i^*} C^n(X;R) \xleftarrow{j^*} C^n(X, A; R) \longleftarrow 0$$

which induces a long exact sequence in cohomology

$$\cdots \longrightarrow H^n(X, A; R) \xrightarrow{H^n(j)} H^n(X;R) \xrightarrow{H^n(i)} H^n(A;R) \xrightarrow{\delta} H^{n+1}(X, A; R) \longrightarrow \cdots$$

Theorem 1.47. *(Excision) Let $Z \subseteq A \subseteq X$ such that the closure of Z is contained in the interior of A. Then, the inclusion $i : (X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces isomorphisms in cohomology*

$$H^n(i) : H^n(X, A; R) \to H^n(X \setminus Z, A \setminus Z; R)$$

for all $n \geq 0$.

Theorem 1.48. *(Mayer-Vietoris) Let $A, B \subseteq X$ such that X is the union of the interiors of A and B. Then, there is a natural long exact sequence called Mayer-Vietoris sequence*

$$\begin{array}{l} \cdots \longrightarrow H^n(X;R) \longrightarrow H^n(A;R) \oplus H^n(B;R) \longrightarrow H^n(A \cap B;R) \\ \qquad \longrightarrow H^{n+1}(X;R) \longrightarrow \cdots \end{array}$$

Remark 1.49. One can analogously define relative Mayer-Vietoris sequences. Let $(X, Y) = (A \cup B, C \cup D)$ with $Y \subseteq X$, $C \subseteq A$, $D \subseteq B$ such that X is the union of the interiors of A and B and Y is the union of the interiors of C and D. Then the relative Mayer-Vietoris sequence is given by

$$\begin{tikzcd} \cdots \arrow[r] & H^n(X,Y;R) \arrow[r] & H^n(A,C;R) \oplus H^n(B,D;R) \arrow[r] & H^n(A\cap B, C \cap D;R) \arrow[dll] \\ & H^{n+1}(X,Y;R) \arrow[r] & \cdots & \end{tikzcd}$$

Cup product

Let us now carry on with new concepts that could not be considered in homology.

Definition 1.50. Consider the affine maps

$$\lambda_{k,l} : \Delta^k \to \Delta^{k+l}$$

$$\rho_{k,l} : \Delta^l \to \Delta^{k+l}$$

determined by

$$\lambda_{k,l}(e_i) = e_i \text{ for } 0 \le i \le k,$$
$$\rho_{k,l}(e_i) = e_{k+i} \text{ for } 0 \le i \le l.$$

Let $\varphi \in C^k(X;R)$, $\psi \in C^l(X;R)$. The *cup product* $\varphi \smile \psi \in C^{k+l}(X;R)$ is the $(k+l)$-cochain determined by

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma\lambda_{k,l})\psi(\sigma\rho_{k,l})$$

for every $(k+l)$-simplex $\sigma : \Delta^{k+l} \to X$.

Lemma 1.51. *Let $\varphi \in C^k(X;R)$, $\psi \in C^l(X;R)$. Then*

$$\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^k \varphi \smile \delta\psi.$$

Remark 1.52. As a consequence of this lemma, the cup product induces a well-defined, associative and distributive map

$$\smile : H^k(X;R) \times H^l(X;R) \to H^{k+l}(X;R)$$

also called *cup product*. One can analogously define a *relative cup product*

$$\smile : H^k(X,A;R) \times H^l(X,B;R) \to H^{k+l}(X, A \cup B;R).$$

Proposition 1.53. *Let $f : X \to Y$ be a continuous map. Then, for all $k, l \ge 0$, we have*

$$H^{k+l}(f)(\alpha \smile \beta) = H^k(f)(\alpha) \smile H^l(f)(\beta)$$

and analogously for the relative case. In fact, this is true at the level of cochains.

Remark 1.54. $H^*(X,A;R) = \bigoplus_{n\ge0} H^n(X,A;R)$ is an associative ring with unit[1] with product defined by $(\sum_i \alpha_i)(\sum_j \beta_j) = \sum_{i,j} \alpha_i \smile \beta_i$. Similarly, we denote by $H^{\prod}(X,A;R)$ the ring that consists of (possibly infinite) formal sums $a_0 + a_1 + \ldots$, where $a_i \in H^i(X,A;R)$. The product on $H^{\prod}(X,A;R)$ is defined in the same way.

Proposition 1.55. *Let $\alpha \in H^k(X,A;R)$ and $\beta \in H^l(X,A;R)$. Then $\alpha \smile \beta = (-1)^{kl}\beta \smile \alpha$.*

[1] or even more: an R-algebra

Definition 1.56. The *cross product* is the map

$$\begin{aligned} H^k(X;R) \times H^l(Y;R) &\to H^{k+l}(X \times Y;R) \\ (a,b) &\mapsto a \times b := H^k(p_X)(a) \smile H^l(p_Y)(b) \end{aligned}$$

where $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are the projection maps. One can analogously define the *relative cross product*

$$H^k(X,A;R) \times H^l(Y,B;R) \to H^{k+l}(X \times Y, A \times Y \cup X \times B;R)$$

Remark 1.57. Let $f_1 : X_1 \to Y_1, f_2 : X_2 \to Y_2$ be continuous and let $p_i^X : X_1 \times X_2 \to X_i, p_i^Y : Y_1 \times Y_2 \to Y_i$ be the projections. Then, the diagram

$$\begin{tikzcd}
C^k(X_1) \times C^l(X_2) \arrow[d, "C^k(p_1^X)\times C^l(p_2^X)"'] & C^k(Y_1) \times C^l(Y_2) \arrow[l, "C^k(f_1)\times C^l(f_2)"'] \arrow[d, "C^k(p_1^Y)\times C^l(p_2^Y)"] \\
C^k(X_1 \times X_2) \times C^l(X_1 \times X_2) \arrow[d, "\smile"'] & C^k(Y_1 \times Y_2) \times C^l(Y_1 \times Y_2) \arrow[l] \arrow[d, "\smile"] \\
C^{k+l}(X_1 \times X_2) & C^{k+l}(Y_1 \times Y_2) \arrow[l, "C^{k+l}(f_1\times f_2)"']
\end{tikzcd}$$

commutes. The middle arrow is $C^k(f_1 \times f_2) \times C^l(f_1 \times f_2)$. The upper square commutes because all maps are induced from continuous maps at the level of topological spaces, where the square clearly commutes. The lower square commutes from Proposition 1.53. As a consequence, we have the following commutative diagram in cohomology:

$$\begin{tikzcd}
H^k(X_1) \times H^l(X_2) \arrow[d, "\times"'] & H^k(Y_1) \times H^l(Y_2) \arrow[l] \arrow[d, "\times"] \\
H^{k+l}(X_1 \times X_2) & H^{k+l}(Y_1 \times Y_2). \arrow[l]
\end{tikzcd}$$

Similarly, the diagram in relative cohomology

$$\begin{tikzcd}
H^k(X_1,A_1) \times H^l(X_2,A_2) \arrow[d, "\times"'] & H^k(Y_1,B_1) \times H^l(Y_2,B_2) \arrow[l] \arrow[d, "\times"] \\
H^{k+l}(X_1 \times X_2, A_1 \times X_2 \cup X_1 \times A_2) & H^{k+l}(Y_1 \times Y_2, B_1 \times Y_2 \cup Y_1 \times B_2) \arrow[l]
\end{tikzcd}$$

also commutes.

Cap product

Definition 1.58. The *cap product* is the R-bilinear map $\frown: C_k(X;R) \times C^l(X;R) \to C_{k-l}(X;R)$ $(k \geq l)$ determined by

$$\sigma \frown \varphi = \varphi(\sigma\lambda_{l,k-l})\sigma\rho_{l,k-l}$$

for any $\varphi \in C^l(X;R)$ and any k-simplex $\sigma : \Delta^k \to X$.

Lemma 1.59. *Let $\sigma \in C_k(X;R)$, $\varphi \in C^l(X;R)$. Then*

$$\partial(\sigma \frown \varphi) = (-1)^l(\partial\sigma \frown \varphi - \sigma \frown \delta\varphi).$$

Remark 1.60. As a consequence of this lemma, the cap product induces a well-defined R-bilinear map

$$\frown : H_k(X;R) \times H^l(X;R) \to H_{k-l}(X;R)$$

called *cap product* as well. Just as with the cup product, there are also *relative cap products*

$$\frown : H_k(X,A;R) \times H^l(X,A;R) \to H_{k-l}(X;R),$$

$$\frown : H_k(X,A;R) \times H^l(X;R) \to H_{k-l}(X,A;R).$$

Proposition 1.61. *The cup product and the cap product are related by*

$$\psi(\alpha \frown \varphi) = (\varphi \smile \psi)(\alpha)$$

for every $\varphi \in C^k(X;R)$, $\psi \in C^l(X;R)$, *and* $\alpha \in C_{k+l}(X;R)$. *In other words, the dual of*

$$C_{k+l}(X;R) \xrightarrow{\frown\varphi} C_l(X;R)$$

is the map

$$C^l(X;R) \xrightarrow{\varphi\smile} C^{k+l}(X;R).$$

Poincaré duality

Theorem 1.62. *(Poincaré duality) Let M be a closed R-orientable m-manifold with fundamental class* $[M] \in H_m(M;R)$. *Then, the map*

$$\begin{aligned} D : H^k(M;R) &\longrightarrow H_{m-k}(M;R) \\ \alpha &\longmapsto [M] \frown \alpha \end{aligned}$$

is an isomorphism for every integer k.

There is also a relative version of this.

Theorem 1.63. *(Relative Poincaré duality) Let M be a compact R-oriented m-manifold with boundary. Then, the cap product by* $[M,\partial M] \in H_m(M,\partial M;R)$ *gives duality isomorphisms*

$$D : H^k(M,\partial M;R) \to H_{m-k}(M;R) \quad \text{and} \quad D' : H^k(M) \to H_{m-k}(M,\partial M;R)$$

for every integer k.

A proof can be found in chapter 21 of [May99].

Chapter 2

Morse theory

2.1 Reeb's Theorem

The material of this section comes mainly from [Mil69]. The goal is to prove the following result.

Theorem 2.1. (Reeb) *Let M be a compact smooth m-manifold and $f : M \to \mathbf{R}$ a smooth map with only two critical points, both of which are non-degenerate. Then M is homeomorphic to the sphere $\mathbb{S}^m$.*

We start by giving the definitions of *critical points* and *non-degenerate critical points*. Unless stated otherwise, M will be an m-dimensional smooth manifold and $f : M \to \mathbf{R}$ will be a smooth map.

Definition 2.2. A point $p \in M$ is called a *critical point* of f if the differential

$$\begin{aligned} d_p f : T_p M &\longrightarrow T_{f(p)}\mathbf{R} \\ v(\bullet) &\longmapsto v(\bullet \circ f) \end{aligned}$$

is zero. This can be stated in terms of local coordinates $(x^1, \dots, x^m)$ in a neighborhood U of p by

$$\frac{\partial f}{\partial x^1}(p) = \dots = \frac{\partial f}{\partial x^m}(p) = 0.$$

The real number $f(p)$ is called a *critical value* of f.

Remark 2.3. Write $M^a = f^{-1}(-\infty, a]$. If a is not a critical value of f, then M^a is a smooth manifold with boundary $f^{-1}(a)$. [1]

Definition 2.4. Let $p \in M$ be a critical point. We define a symmetric bilinear map

$$f_{**} : T_p M \times T_p M \to \mathbf{R}$$

by the following steps:

(i) Take $v, w \in T_p M$.

(ii) Choose extensions to vector fields $\tilde{v}, \widetilde{w}$. [2]

(iii) Take $f_{**}(v, w) = \tilde{v}_p(\widetilde{w}(f))$.

[1]See Proposition 5.47 in [Lee12]. This fact may be seen as a generalization of the *regular value theorem*.

[2]Here vector fields are linear maps $X : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$ that satisfy the Leibniz rule $X(fg) = X(f)g + fX(g)$.

One can check that this is well-defined and that for local coordinates $(x^1, \dots, x^m)$ in a neighborhood U of p, the Gram matrix of f_{**} is

$$\left(\frac{\partial^2 f}{\partial x^i \partial x^j}(p)\right)$$

with respect to the basis $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^m}\Big|_p \right\}$.

We say that the point p is *non-degenerate* if f_{**} is non-degenerate or, equivalently, if the previous matrix is invertible.

We define the *index* of f at p as the maximal dimension of a subspace of T_pM on which f_{**} is negative definite.

Still before giving a proof of the theorem we state and show a bunch of lemmas.

Lemma 2.5. *Let $V \subseteq \mathbf{R}^m$ be a convex neighborhood of 0 and $f \in \mathscr{C}^\infty(V)$ with $f(0) = 0$. Then*

$$f(x_1, \dots, x_m) = \sum_{i=1}^m x_i g_i(x_1, \dots, x_m)$$

for some $g_i \in \mathscr{C}^\infty(V)$ with $g_i(0) = \frac{\partial f}{\partial x_i}(0)$.

Proof. Since V is convex, all points of the form $(tx_1, \dots, tx_m)$ for $t \in [0,1]$ lie in V (of course as long as $(x_1, \dots, x_m) \in V$). Now by the fundamental theorem of calculus and the chain rule, we have

$$f(x_1, \dots, x_m) = \int_0^1 \frac{df(tx_1, \dots, tx_m)}{dt} dt = \int_0^1 \sum_{i=1}^m \frac{\partial f}{\partial x_i}(tx_1, \dots, tx_m) x_i dt = \sum_{i=1}^m x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \dots, tx_m) dt.$$

It suffices then to define $g_i(x_1, \dots, x_m) = \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \dots, tx_m) dt$. □

Lemma 2.6. (Morse) *Let $p \in M$ be a non-degenerate critical point of f. Then there are local coordinates $(y^1, \dots, y^m)$ in a neighborhood U of p such that $y^i(p) = 0$ and*

$$f = f(p) - (y^1)^2 - \dots - (y^\lambda)^2 + (y^{\lambda+1})^2 + \dots + (y^m)^2$$

throughout U, where λ is the index of f at p.

Proof. We divide the proof into two steps. We first show that if such an expression exists, then λ is the index of f at p. Secondly, we show that for suitable local coordinates $(y^1, \dots, y^m)$ such an expression holds.

Assume there are local coordinates $(z^1, \dots, z^m)$ in a neighborhood V of p such that

$$f(q) = f(p) - (z^1(q))^2 - \dots - (z^\lambda(q))^2 + (z^{\lambda+1}(q))^2 + \dots (z^m(q))^2.$$

Then, with respect to the basis $\left\{ \frac{\partial}{\partial z^1}\Big|_p, \dots, \frac{\partial}{\partial z^m}\Big|_p \right\}$, f_{**} has Gram matrix

$$\begin{pmatrix} -2 & & & & & \\ & \ddots^{(\lambda)} & & & & \\ & & -2 & & & \\ & & & 2 & & \\ & & & & \ddots^{(m-\lambda)} & \\ & & & & & 2 \end{pmatrix}.$$

So there is a subspace of T_pM of dimension λ on which f_{**} is negative definite (namely, the subspace $\langle \frac{\partial}{\partial z^1}\big|_p, \dots, \frac{\partial}{\partial z^\lambda}\big|_p\rangle$). If there were a subspace of dimension greater than λ on which f_{**} were negative definite, then it would intersect $\langle \frac{\partial}{\partial z^{\lambda+1}}\big|_p, \dots, \frac{\partial}{\partial z^m}\big|_p\rangle$. But f_{**} is positive definite on this last subspace, so we reached a contradiction. Thus, λ is the index of f at p. This proves the first claim.

For the second claim, start with local coordinates $\varphi = (x^1, \dots, x^m)$ in a convex neighborhood V of p. Without loss of generality we can assume that $\varphi(p) = 0$ (otherwise perform a translation) and that $f(p) = 0$. From Lemma 2.5, we have

$$(f \circ \varphi^{-1})(x_1, \dots, x_m) = \sum_{j=1}^{m} x_j g_j(x_1, \dots, x_m)$$

and

$$g_j(0) = \frac{\partial(f \circ \varphi^{-1})}{\partial x_j}(0) = \frac{\partial f}{\partial x^j}(p) = 0$$

since p is a critical point. Therefore we can apply again Lemma 2.5 to the functions g_j, which gives

$$g_j(x_1, \dots, x_m) = \sum_{i=1}^{m} x_i h_{ij}(x_1, \dots, x_m).$$

Substituting into the previous expression for $(f \circ \varphi^{-1})$ we have

$$(f \circ \varphi^{-1})(x_1, \dots, x_m) = \sum_{i,j=1}^{m} x_i x_j h_{ij}(x_1, \dots, x_m).$$

We can further assume that $h_{ij} = h_{ji}$ (just replace h_{ij} by $\frac{1}{2}(h_{ij} + h_{ji})$ and the same expression for $f \circ \varphi^{-1}$ will hold). A (rather long) computation shows that

$$(h_{ij}(0)) = \left(\frac{1}{2}\frac{\partial^2(f \circ \varphi^{-1})}{\partial x_i \partial x_j}(0)\right) = \left(\frac{1}{2}\frac{\partial^2 f}{\partial x^i \partial x^j}(p)\right).$$

Since p is non-degenerate, the matrix $(h_{ij}(0))_{1 \le i,j \le m}$ is invertible (hence non-zero). The rest of the proof just mimics the usual diagonalization proof of symmetric bilinear forms. We proceed by induction. Assume there are local coordinates $u = (u^1, \dots, u^m)$ in a neighborhood U_1 of p such that $u(p) = 0$ and

$$(f \circ u^{-1})(u_1, \dots, u_m) = \pm(u_1)^2 \pm \dots \pm (u_{r-1})^2 + \sum_{i,j \ge r}^{m} u_i u_j H_{ij}(u_1, \dots, u_m)$$

for $(u_1, \dots, u_m) \in u(U_1)$, where the matrices $(H_{ij}(u_1, \dots, u_m))$ are symmetric and $(H_{ij}(0))_{r \le i,j \le m}$ is invertible. Without loss of generality, we can assume that $H_{rr}(0) \neq 0$. [3] Choose a (possibly) smaller neighborhood $U_2 \subseteq U_1$ of p so that $H_{rr}(u_1, \dots, u_m) \neq 0$ for $(u_1, \dots, u_m) \in u(U_2)$. Define new local coordinates $v = (v^1, \dots, v^m)$ by the equations

$$v^i = u^i, \text{ for } i \neq r$$

$$(v^r \circ u^{-1})(u_1, \dots, u_m) = \sqrt{|H_{rr}(u_1, \dots, u_m)|}\left[u_r + \sum_{i>r} u_i \frac{H_{ir}(u_1, \dots, u_m)}{H_{rr}(u_1, \dots, u_m)}\right].$$

By the inverse function theorem, there is a (possibly) smaller neighborhood $U_3 \subseteq U_2$ of p on which $(v \circ u^{-1})$ is a diffeomorphism. On this new neighborhood U_3, $v = (v^1, \dots, v^m)$ are then indeed local coordinates. Now easy computations yield

$$(f \circ v^{-1})(v_1, \dots, v_m) = \sum_{i \le r}(v_i)^2 + \sum_{i,j>r} v_i v_j H'_{ij}(v_1, \dots, v_m)$$

[3] Just do a suitable linear (therefore smooth) change in the last $n - r + 1$ coordinates (this is possible because $(H_{ij}(0))_{1 \le i,j \le r}$ is invertible).

for $(v_1, \dots, v_m) \in v(U_3)$, where

$$((H'_{ij} \circ (v \circ u^{-1}))(u_1, \dots, u_m)) = \left(H_{ij}(u_1, \dots, u_m) - \frac{H_{ir}(u_1, \dots, u_m)H_{jr}(u_1, \dots, u_m)}{H_{rr}(u_1, \dots, u_m)}\right)$$

is symmetric and $(H'_{ij}(0))_{r+1 \le i,j \le m}$ is invertible, which completes the inductive step and finishes the proof. □

Definition 2.7. A *1-parameter group of diffeomorphisms* of M is a smooth map $\varphi : \mathbf{R} \times M \to M$ such that

(i) for every $t \in \mathbf{R}$, the map $\varphi_t = \varphi(t, \bullet) : M \to M$ is a diffeomorphism.

(ii) for every $t, s \in \mathbf{R}$, $\varphi_{t+s} = \varphi_t \circ \varphi_s$.

Given such a map φ, define a smooth vector field X on M by

$$X_q(f) = \lim_{h \to 0} \frac{f(\varphi_h(q)) - f(q)}{h} = \left.\frac{df(\varphi_t(q))}{dt}\right|_{t=0} = \left.\frac{d\varphi_t(q)}{dt}\right|_{t=0}(f)$$

for every $q \in M$ and $f \in \mathscr{C}^\infty(M)$. The vector field X is said to *generate* the group φ.

Lemma 2.8. *A smooth vector field X on M which vanishes outside of a compact set $K \subseteq M$ generates a unique 1-parameter group of diffeomorphisms of M.*

Proof. Let φ be a 1-parameter group of diffeomorphisms with generator X. Notice that

$$X_{\varphi_s(q)}(f) = \left.\frac{d\varphi_t(\varphi_s(q))}{dt}\right|_{t=0}(f) = \left.\frac{d\varphi_{t+s}(q)}{dt}\right|_{t=0}(f) = \left.\frac{d\varphi_t(q)}{dt}\right|_{t=s}(f).$$

Therefore, φ satisfies the ODEs

$$\frac{d\varphi_t(q)}{dt} = X_{\varphi_t(q)} \text{ with initial condition } \varphi_0(q) = q. \quad (*)$$

It is a standard result in Differential Geometry that these ODEs have a locally unique maximal solution that is smooth on t and on the initial condition $q \in M$. [4] More precisely, for each point $p \in M$ there is a neighborhood U_p of p and a real number $\varepsilon_p > 0$ so that $(*)$ has a unique smooth solution $\varphi_t(q)$ for $q \in U_p$ and $|t| < \varepsilon_p$. This proves the uniqueness part of the lemma.

For the existence part it suffices to show that there exists a 1-parameter group of diffeomorphisms $\varphi : \mathbf{R} \times M \to M$ satisfying $(*)$. Cover K with a finite number of neighborhoods $U_{p_1}, \dots, U_{p_n}$ defined as above. Let $\varepsilon_0 = \min\{\varepsilon_{p_1}, \dots, \varepsilon_{p_n}\}$ and set $\varphi_t(q) = q$ for $q \notin K, t \in \mathbf{R}$. Then $(*)$ has a smooth solution $\varphi_t(q)$ for $q \in M$ and $|t| < \varepsilon_0$. Because of local uniqueness, we also have $\varphi_{t+s} = \varphi_t \circ \varphi_s$ when $|t|, |s|, |t+s| < \varepsilon_0$. In particular, each φ_t is a diffeomorphism.

Now we just need to define φ_t for $|t| \ge \varepsilon_0$. Write t as $k(\varepsilon_0/2) + r$ with $k \in \mathbf{Z}$ and $|r| < \varepsilon_0/2$. If $k \ge 0$, set

$$\varphi_t = (\varphi_{\varepsilon_0/2})^k \circ \varphi_r.$$

If $k < 0$, set

$$\varphi_t = (\varphi_{-\varepsilon_0/2})^{-k} \circ \varphi_r.$$

This is well-defined, smooth, satisfies $(*)$ and $\varphi_{t+s} = \varphi_t \circ \varphi_s$ for all $t, s \in \mathbf{R}$. □

[4] One can see, for example, Theorem 9.12 in [Lee12].

Theorem 2.9. *Let $a < b$ be real numbers such that $f^{-1}[a,b] \subseteq M$ is compact and contains no critical points of f. Then M^a is diffeomorphic to M^b and M^a is a deformation retract of M^b.*

Proof. Choose a Riemannian metric $g(\bullet,\bullet) = \langle \bullet,\bullet \rangle$ on M.[5] Consider the vector field $\operatorname{grad} f$ determined by

$$\langle v, (\operatorname{grad} f)_p \rangle = v(f)$$

for every $p \in M$ and $v \in T_pM$. Notice that $(\operatorname{grad} f)_p = 0$ if and only if p is a critical point of f.

Choose a smooth function $\rho : M \to \mathbf{R}$ which equals $1/\langle \operatorname{grad} f, \operatorname{grad} f \rangle$ throughout the compact set $f^{-1}[a,b]$ and vanishes outside of a compact neighborhood of $f^{-1}[a,b]$. Then the vector field

$$X_q = \rho(q)(\operatorname{grad} f)_q$$

satisfies the conditions of Lemma 2.8, therefore generates a 1-parameter group of diffeomorphisms $\varphi_t : M \to M$. For every $q \in M$ consider the function $\Phi_q : \mathbf{R} \to \mathbf{R}$ given by $\Phi_q(t) = f(\varphi_t(q))$. Notice that, if $\varphi_t(q) \in f^{-1}[a,b]$, then

$$\frac{d\Phi_q(t)}{dt} = \frac{df(\varphi_t(q))}{dt} = \left\langle \frac{d\varphi_t(q)}{dt}, (\operatorname{grad} f)_{\varphi_t(q)} \right\rangle = \langle X_{\varphi_t(q)}, (\operatorname{grad} f)_{\varphi_t(q)} \rangle = 1.$$

Thus the map $\Phi_q(t)$ has derivative $+1$ when $\Phi_q(t) \in [a,b]$. Now consider the diffeomorphism $\varphi_{b-a} : M \to M$. We claim that it restricts to a diffeomorphism $M^a \to M^b$. Indeed:

- If $q \in M^a$, then $\Phi_q(0) = f(\varphi_0(q)) = f(q) \in (-\infty, a]$. Now, if we increase t from 0 to $b-a$, then $\Phi_q(t)$ varies continuously and if $\Phi_q(t_0)$ reaches a at some point t_0, then $\Phi_q(t)$ carries on increasing with constant slope $+1$, so it can never exceed b. Hence, $f(\varphi_{b-a}(q)) = \Phi_q(b-a) \le b$ and $\varphi_{b-a}(q) \in M^b$.

- If $q \in M^b$ a similar argument shows that $p = \varphi_{a-b}(q) \in M^a$, so $\varphi_{b-a}(p) = q$.

This finishes the first part of the proof.

For the second part, define a homotopy $r : [0,1] \times M^b \to M^b$ by

$$r_t(q) = r(t,q) = \begin{cases} q, & f(q) \le a. \\ \varphi_{t(a-f(q))}(q), & a \le f(q) \le b. \end{cases}$$

By an argument with slopes similar to the one we already used, one can see that r is well-defined (i.e. has image in M^b) and that r_1 takes M^b to M^a. Since r is continuous on the closed sets $[0,1] \times f^{-1}(-\infty, a]$, $[0,1] \times f^{-1}[a,b]$ separately (and is well-defined on the intersection), it is continuous on $[0,1] \times M^b$. Because of this and the facts that $r_0 = \mathrm{id}_{M^b}$, $r_1 : M^b \to M^a$ and $r_1|_{M^a} = \mathrm{id}_{M^a}$, we have just proven that M^a is a deformation retraction of M^b. □

Now we are ready to give a proof of Reeb's Theorem.

Proof. (of Theorem 2.1) Since M is compact, $f(M)$ has a minimum $a \in \mathbf{R}$ and a maximum $b \in \mathbf{R}$. Take $p \in M$ with $f(p) = a$ and a chart (U, φ) about p. Then $(f \circ \varphi^{-1}) : \varphi(U) \subseteq \mathbf{R}^m \to \mathbf{R}$ reaches its minimum at $\varphi(p)$, so by standard results of differentiable analysis, $\varphi(p)$ must be a critical point of $(f \circ \varphi^{-1})$, i.e. p must be a critical point of f. Similarly for a point $q \in M$ with $f(q) = b$. Therefore, the two critical points of f are p and q.
By Lemma 2.6 we can write

$$f = a + (y^1)^2 + \ldots + (y^m)^2 \text{ in a neighborhood } U_p \text{ of } p$$

[5] This can always be done. See, for example, Proposition 13.3 in [Lee12].

with no minus signs because a is the minimum of f. Similarly,

$$f = b - (y^1)^2 - \ldots - (y^m)^2 \text{ in a neighborhood } U_q \text{ of } q$$

Choose $\varepsilon > 0$ small enough so that $M^{a+\varepsilon} = f^{-1}[a, a+\varepsilon]$ lies in U_p and $f^{-1}[b-\varepsilon, b]$ lies in U_q. By the previous expressions for f, $M^{a+\varepsilon}$ and $f^{-1}[b-\varepsilon, b]$ are homeomorphic to closed m-cells. Furthermore, by Theorem 2.9, $M^{a+\varepsilon}$ is homeomorphic to $M^{b-\varepsilon}$. Thus M is the union of two closed m-cells $M^{b-\varepsilon} = f^{-1}[a, b-\varepsilon]$ and $f^{-1}[b-\varepsilon, b]$ glued along their common boundary. This is a well-known description of the sphere $\mathbb{S}^m$ as a CW-complex. In particular, M is homeomorphic to $\mathbb{S}^m$. $\square$

2.2 Construction of M_k^7

In this section we give a construction of 7-manifolds M_k^7 for $k \in \mathbf{Z}$ odd and show that they are all homeomorphic to the sphere $\mathbb{S}^7$ just as Milnor did in [Mil56]. Later on we will see that some of these manifolds are in fact exotic spheres.

In order to define such manifolds, we need the following lemma.

Lemma 2.10. *Let M_1, M_2 be smooth m-dimensional manifolds, $U_i \subseteq M_i$ be open subsets and $g : U_1 \to U_2$ be a diffeomorphism satisfying that every point $x \in \partial U_1$*[6] *has a neighborhood $V_1 \subseteq M_1$ such that the closure of $g(V_1 \cap U_1)$ in M_2 is contained in U_2. Then, the quotient $\widehat{M} = \frac{M_1 \sqcup M_2}{u \sim g(u)}$ has a natural smooth structure of dimension m.*

Proof. Denote $M = M_1 \sqcup M_2$. We start by claiming that the quotient map $\pi : M \to \widehat{M}$ is open and that the restrictions $\pi_{|M_1}, \pi_{|M_2}$ are homeomorphisms onto their respective images. Indeed, for an open subset $V_1 \subseteq M_1$, we have $\pi^{-1}(\pi(V_1)) = V_1 \cup g(V_1 \cap U_1)$ which is open in M. Similarly for an open subset $V_2 \subseteq M_2$. Since $\pi_{|M_1}$ and $\pi_{|M_2}$ are injective, the second claim also follows. As a consequence, $\widehat{M}$ is locally Euclidean and second countable.

Now we check that the space $\widehat{M}$ is Hausdorff. Let $p, q \in \widehat{M}$, $p \neq q$. Choose representatives $x, y \in M$ of p, q, respectively. If x and y belong to the same M_i, the result is clear because $\pi_{|M_i}$ are homeomorphisms. Without loss of generality, assume that $x \in M_1$ and $y \in M_2$ and distinguish cases $x \in U_1, M_1 \setminus \overline{U}_1, \partial U_1, y \in U_2, M_1 \setminus \overline{U}_2, \partial U_2$. Most of these are easily solved, so here we only deal with the problematic case $x \in \partial U_1, y \in \partial U_2$. Take $V_1 \subseteq M_1$ given by the hypothesis of the lemma. Then, $V_2 = M_2 \setminus \overline{g(V_1 \cap U_1)}$ is a neighborhood of y and $\pi(V_1) \cap \pi(V_2) = \emptyset$, as wanted.

Finally, consider the charts on $\widehat{M}$ of the form $(\pi(V), \varphi \circ \pi^{-1}_{|V})$, where (V, φ) is a chart on M_1 or on M_2. These clearly cover $\widehat{M}$, so for them to form an atlas we just need to check compatibility. Let (V, φ) and (W, ψ) be charts on M_1 or on M_2. If both are charts on the same M_i, then

$$(\varphi \circ \pi^{-1}_{|V}) \circ (\psi \circ \pi^{-1}_{|W})^{-1} = \varphi \circ \psi^{-1}$$

is smooth. If (V, φ) is on M_1 and (W, ψ) is on M_2 (the other way around is done similarly), then

$$(\varphi \circ \pi^{-1}_{|V}) \circ (\psi \circ \pi^{-1}_{|W})^{-1} = \varphi \circ \pi^{-1}_{|V} \circ \pi_{|W} \circ \psi^{-1} = \varphi \circ \pi^{-1}_{|V} \circ \pi_{|g^{-1}(W)} \circ g^{-1} \circ \psi^{-1} = \varphi \circ g^{-1} \circ \psi^{-1}$$

is also smooth. $\square$

We now identify the space $\mathbf{R}^4$ with the ring of quatertions $\mathbf{H}$ via

$$(a, b, c, d) \mapsto a + bi + cj + dk$$

[6] The notation ∂U here is to be interpreted as the space $\overline{U} \setminus \mathring{U}$. It is thus not to be confused with the boundary of a manifold.

where $i^2 = j^2 = k^2 = ijk = -1$. We also identify the sphere $\mathbb{S}^3$ with the unit quaternions

$$\{q = a + bi + cj + dk \in \mathbf{H} : \|q\|^2 = a^2 + b^2 + c^2 + d^2 = 1\}.$$

For a quaternion $q = a + bi + cj + dk$ we denote its conjugate by $\bar{q} = a - bi - cj - dk$.

Lemma 2.11. *Let k be an odd integer. Let h, j be the integers uniquely determined by $h + j = 1, h - j = k$. Then, the map*

$$\begin{aligned} g : (\mathbf{R}^4 \setminus \{0\}) \times \mathbb{S}^3 &\longrightarrow (\mathbf{R}^4 \setminus \{0\}) \times \mathbb{S}^3 \\ (u, v) &\longmapsto (u', v') = \left(\frac{u}{\|u\|^2}, \frac{u^h v u^j}{\|u\|^{h+j}} \right) \end{aligned}$$

is a diffeomorphism. Furthermore, the 7-manifold $M_1 = M_2 = \mathbf{R}^4 \times \mathbb{S}^3$ with $U_1 = U_2 = (\mathbf{R}^4 \setminus \{0\}) \times \mathbb{S}^3$ and the diffeomorphism g satisfy the conditions of Lemma 2.10. The resulting quotient manifold is denoted by M_k^7.

Proof. Injectivity and surjectivity of g can be easily checked. Smoothness is simply checked by noting that g can be extended to a map $(\mathbf{R}^4 \setminus \{0\}) \times \mathbf{R}^4 \longrightarrow (\mathbf{R}^4 \setminus \{0\}) \times \mathbf{R}^4$ that is clearly smooth. Since g^{-1} has a similar expression, namely

$$g^{-1}(u', v') = \left(\frac{u'}{\|u'\|^2}, \frac{(\overline{u'})^h v' (\overline{u'})^j}{\|u'\|} \right),$$

the same argument applies. Finally, if $x \in \partial U_1$, then $x = (0, v)$ and we can just choose $V_1 = B_1(0) \times \mathbb{S}^3$, which gives $\overline{g(V_1 \cap U_1)} \subseteq (\mathbf{R}^4 \setminus B_1(0)) \times \mathbb{S}^3 \subseteq U_1$. $\square$

Lemma 2.12. *Define new coordinates $(u'', v') = (u'(v')^{-1}, v')$. The map $f : M_k^7 \to \mathbf{R}$ given by*

$$f(x) = \begin{cases} \dfrac{\Re(v)}{\sqrt{1 + \|u\|^2}} & \text{if } x = (u, v), \\ \dfrac{\Re(u'')}{\sqrt{1 + \|u''\|^2}} & \text{if } x = (u', v'). \end{cases}$$

is well-defined, smooth and has only two critical points, both of which are non-degenerate.

Proof. We start by showing that $f(u, v) = f(u', v')$ for $(u', v') = g(u, v)$. We have $u'' = u'(v')^{-1} = \frac{u}{\|u\|^2} \|u\| u^{-j} v^{-1} u^{-h} = \frac{u^h \bar{v} u^{-h}}{\|u\|}$, so $\|u''\| = \frac{1}{\|u\|}$. Hence

$$\frac{\Re(u'')}{\sqrt{1 + \|u''\|^2}} = \frac{\Re\left(\frac{u^h \bar{v} u^{-h}}{\|u\|}\right)}{\sqrt{1 + \|u\|^{-2}}} = \frac{\Re(u^h \bar{v} u^{-h})}{\sqrt{1 + \|u\|^2}} = \frac{\Re(v)}{\sqrt{1 + \|u\|^2}}.$$

The last equality follows from

$$\begin{aligned} \Re(u^h \bar{v} u^{-h}) &= \frac{1}{2} \left(u^h \bar{v} u^{-h} + \bar{u}^{-h} v \bar{u}^h \right) = \frac{1}{2} \left(u^h \bar{v} u^{-h} + \frac{\|u\|^{-2h}}{u^{-h}} v \frac{\|u\|^{2h}}{u^h} \right) \\ &= u^h \Re(v) u^{-h} = \Re(v). \end{aligned}$$

This proves that f is well-defined.

Denote the charts of M_i by $(U^i_\pm \varphi^i_\pm)$. Notice that the four charts of M^7_k of the form $(\pi(U^i_\pm), \varphi^i_\pm \circ \pi^{-1}_{|U^i_\pm})$ already cover M^7_k. For $(w, s) \in \mathbf{R}^4 \times \mathbf{R}^3$, we have

$$(f \circ \pi_{|U^1_\pm} \circ (\varphi^1_\pm)^{-1})(w, s) = f\left(w, \pm\frac{1-\sigma(s)}{1+\sigma(s)}, \dots\right) = \pm\frac{1-\sigma(s)}{(1+\sigma(s))\sqrt{1+\|w\|^2}}.$$

Writing $\Gamma(s) = \frac{1-\sigma(s)}{1+\sigma(s)}, \chi(w) = \frac{1}{\sqrt{1+\|w\|^2}}$, the Jacobian matrix of this map is

$$\mp\chi(w)\begin{pmatrix} \frac{\Gamma(s)w_1}{1+\|w\|^2} & \frac{\Gamma(s)w_2}{1+\|w\|^2} & \frac{\Gamma(s)w_3}{1+\|w\|^2} & \frac{\Gamma(s)w_4}{1+\|w\|^2} & \frac{4}{(1+\sigma(s))^2}s_1 & \frac{4}{(1+\sigma(s))^2}s_2 & \frac{4}{(1+\sigma(s))^2}s_3 \end{pmatrix}.$$

Notice that this matrix is zero only when $(w, s) = (0, 0)$. Hence, the only critical points of f on $\pi(M_1)$ are $(u, v) = (0, \pm 1)$. One can easily compute the Hessian matrix and evaluate it at $(w, s) = (0, 0)$ to obtain a diagonal matrix in which every diagonal element is non-zero. This proves that the points $(u, v) = (0, \pm 1)$ are non-degenerate critical points. We still need to check that there are no critical points on $\pi(M_2)$. For $(w, s) \in \mathbf{R}^4 \times \mathbf{R}^3$, we have

$$(f \circ \pi_{|U^2_\pm} \circ (\varphi^2_\pm)^{-1})(w, s) = \chi(w)\Re(w\overline{v'}),$$

where

$$v' = \left(\pm\Gamma(s), \frac{2s_1}{1+\sigma(s)}, \frac{2s_2}{1+\sigma(s)}, \frac{2s_3}{1+\sigma(s)}\right)$$

and hence

$$\Re(w\overline{v'}) = \pm w_1\Gamma(s) + w_2\frac{2s_1}{1+\sigma(s)} + w_3\frac{2s_2}{1+\sigma(s)} + w_4\frac{2s_3}{1+\sigma(s)},$$

$$(f \circ \pi_{|U^2_\pm} \circ (\varphi^2_\pm)^{-1})(w, s) = \frac{\chi(w)}{1+\sigma(s)}\left[\pm w_1(1-\sigma(s)) + 2w_2s_1 + 2w_3s_2 + 2w_4s_3\right].$$

Since points of the from (u', v') with $u' \neq 0$ are identified with points (u, v) with $u \neq 0$, it suffices to compute the Jacobian matrix and evaluate it at $w = u' = 0$. Doing so, we obtain

$$\begin{pmatrix} \pm\Gamma(s) & \frac{2s_1}{1+\sigma(s)} & \frac{2s_2}{1+\sigma(s)} & \frac{2s_3}{1+\sigma(s)} & \dots \end{pmatrix}$$

and we already see that the first four terms cannot simultaneously vanish. □

As M^7_k is clearly compact, these lemmas and Reeb's theorem 2.1 imply what we wanted.

Corollary 2.13. *The manifolds M^7_k are homeomorphic to $\mathbb{S}^7$.*

Chapter 3

Characteristic classes

We switch now to a completely different topic. In this chapter, the concept of vector bundle is introduced in Section 3.1. The ultimate goal is to define several characteristic classes, which is done later in Sections 3.2-3.4. Finally, we state and sketch a proof of the Hirzebruch signature theorem in Section 3.5.

The main reference for the whole chapter is [MS74].

3.1 Vector bundles

Definition 3.1. Let B be a topological space. A *(real) vector bundle* ξ over B consists of

(i) the given space B, which will be referred to as the *base space*,

(ii) a topological space $E = E(\xi)$ called the *total space*,

(iii) a continuous map $\pi : E \longrightarrow B$ called the *projection map* and

(iv) a real vector space structure on the sets $\pi^{-1}(b)$ for every $b \in B$.

Furthermore, the condition of local triviality must be satisfied. Namely, every $b \in B$ has a neighborhood $U \subseteq B$, an integer $n \geq 0$ and a homeomorphism

$$h : U \times \mathbf{R}^n \longrightarrow \pi^{-1}(U)$$

so that for every $\bar{b} \in U$, the map

$$\begin{aligned} h_{\bar{b}} = h(\bar{b}, \bullet) : \mathbf{R}^n &\longrightarrow \pi^{-1}(\bar{b}) \\ x &\longmapsto h(\bar{b}, x) \end{aligned}$$

is an isomorphism of $\mathbf{R}$-vector spaces. Such a pair (U, h) is called *local coordinate system* for ξ about b. The vector space $\pi^{-1}(b)$ is also denoted by $F_b(\xi)$ (or simply F_b) and is called *fiber* over b. If U can be chosen to be the entire base space B, then ξ will be called a *trivial bundle*.

Remark 3.2. Because of the local triviality property, n is a locally constant function of b. In our setting, B will always be connected. Hence n will always be a global constant and its value will be specified by saying that ξ is an $\mathbf{R}^n$*-bundle* over B.

Remark 3.3. A *smooth (real) vector bundle* is a (real) vector bundle for which B and E are smooth manifolds, π is a smooth map and the local coordinate systems (U, h) can be chosen so that h is a diffeomorphism.

Definition 3.4. Let ξ and η be $\mathbf{R}^n$-bundles. A *bundle map* $\xi \longrightarrow \eta$ is a continuous function $f : E(\xi) \longrightarrow E(\eta)$ that maps each fiber $F_b(\xi)$ isomorphically onto one of the fibers $F_{b'}(\eta)$ as $\mathbf{R}$-vector spaces. We set $\bar{f}(b) = b'$. The map $\bar{f} : B(\xi) \longrightarrow B(\eta)$ is easily seen to be continuous. We also say that $\bar{f}$ is covered by f.

Definition 3.5. Two vector bundles ξ and η over the same base space B are isomorphic (written $\xi \cong \eta$) if there is a bundle map $\xi \to \eta$ that is a homeomorphism $E(\xi) \to E(\eta)$ and covers the identity $B \to B$. Notice that ξ is trivial if, and only if, it is isomorphic to $E = \mathbf{R}^n \times B$ with projection $\pi(x, b) = b$.

Contructing vector bundles

We now turn our attention to briefly describe how to construct new vector bundles out of old ones.

Definition 3.6. Let ξ be a vector bundle over B with projection $\pi : E \to B$ and let $B_1 \subseteq B$. Then, the restriction $\pi_{|E_1} : E_1 \to B_1$ of π to $E_1 = \pi^{-1}(B_1)$ gives rise to a new vector bundle over B_1 called the *restriction* of ξ to B_1 and is denoted by $\xi_{|B_1}$. The vector space structure on each fiber $F_b(\xi_{|B_1})$ is the same as the given structure on $F_b(\xi)$.

Definition 3.7. Let ξ be a vector bundle over B with projection $\pi : E \to B$. Let further B_1 be an arbitrary topological space and $f : B_1 \to B$ a continuous map. The *induced bundle* $f^*\xi$ over B_1 is constructed as follows. Take its total space to be $E_1 = \{(b, e) \in B_1 \times E : f(b) = \pi(e)\}$ and projection $\pi_1 : E_1 \to B_1$ defined by $(b, e) \mapsto b$. The vector space structure on each fiber is the obvious one.

Definition 3.8. Let ξ_1, ξ_2 be two vector bundles with projections $\pi_i : E_i \to B_i$. The *Cartesian product bundle* $\xi_1 \times \xi_2$ is the bundle with projection $\pi = \pi_1 \times \pi_2 : E_1 \times E_2 \to B_1 \times B_2$ and obvious vector space structure on each fiber.

Definition 3.9. Let ξ_1, ξ_2 be two vector bundles. The *Whitney sum bundle* $\xi_1 \oplus \xi_2$ is defined as $d^*(\xi_1 \times \xi_2)$, where $d : B \to B \times B$ is the diagonal embedding.

Remark 3.10. The motivation behind this notation comes from the fact that the fibers $F_b(\xi_1 \oplus \xi_2)$ are canonically isomorphic to the direct sum $F_b(\xi_1) \oplus F_b(\xi_2)$.

For the next construction, we need first to define what a Euclidean metric on a vector bundle is.

Definition 3.11. A *Euclidean vector bundle* is a vector bundle ξ equipped with a continuous function $\mu : E(\xi) \to \mathbf{R}$ that is positive definite and quadratic on each fiber. Using a partition of unity, it can be shown that such a function always exists if the base space is paracompact.

Definition 3.12. Let η be a Euclidean vector bundle over B and let $\xi \subset \eta$ be a sub-bundle, i.e. a vector bundle over B whose fibers are vector subspaces of the fibers of η. The *orthogonal complement* of ξ in η is the sub-bundle $\xi^\perp$ of η whose fibers are the orthogonal complements $F_b(\xi)^\perp$ of $F_b(\xi)$ in $F_b(\eta)$.

Remark 3.13. Since the map continuous map $f : E(\xi \oplus \xi^\perp) \to E(\eta)$, $(b, e_1, e_2) \mapsto e_1 + e_2$ defines an isomorphism on each fiber $= F_b(\eta) = F_b(\xi) \oplus F_b(\xi^\perp)$, it follows that $\eta \cong \xi \oplus \xi^\perp$. In words, any Euclidean vector bundle can be decomposed into a Whitney sum of orthogonal sub-bundles.

Many of these constructions can be generalized as follows. Denote the category of finite dimensional vector spaces and isomorphisims by $\mathcal{U}$. Let $T : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ be a functor. Notice that any finite dimensional vector space can be naturally topologized. In particular, for any finite dimensional vector spaces E, F, the set $\mathrm{Iso}(E, F)$, being a subset of $\mathrm{Hom}(E, F)$, has a natural topology. We further assume that $T(f, g)$ varies continuously on f and g in this sense. Given vector bundles ξ_1, ξ_2 over B, we can then naturally construct a new vector bundle $\xi = T(\xi_1, \xi_2)$ whose fibers are $F_b(\xi) = T(F_b(\xi_1), F_b(\xi_2))$. Details on these assertions can be found in [MS74] pp. 31-34.

Orientation of vector bundles

Now, let V be an n-dimensional real vector space. As already said, V has a natural topology, which allows us to define an R-orientation of V as a choice of a generator of $H_n(V|0;R)$. Even more, because of the universal coefficient theorem for cohomology 1.39, the map h *loc. cit.* defines an isomorphism $H^n(V|0;R) \cong \mathrm{Hom}_R(H_n(V|0;R),R)$. Since $H_n(V|0;R) \cong R$ is free, for every generator μ of $H_n(V|0;R)$ there is a unique cohomology class $u \in H^n(V|0;R)$ such that $h(u)(\mu) = \langle u,\mu\rangle = 1$. One can thus also define an R-orientation of V as a choice of a generator of $H^n(V|0;R)$. This last definition is the one we are going to use in the following sections unless otherwise specified.

Given a subset $E' \subseteq E$ of the total space of a bundle, we will denote the set of nonzero elements in E' by E'_0. In other words, $E'_0 = E' \setminus \{p \in E' : p = 0 \text{ in some fiber } F\}$.

Definition 3.14. An *R-orientation for a vector bundle ξ* over B is a function that assigns an R-orientation to each fiber, i.e. a choice of a generator $u_F \in H^n(F,F_0;R)$ for every fiber F. This function is also required to satisfy the following local compatibility condition: for every $b \in B$ there is a neighborhood N of b and a cohomology class

$$u \in H^n(\pi^{-1}(N), \pi^{-1}(N)_0; R)$$

so that, for each fiber F over N, the image of u by the restriction homomorphism

$$\begin{aligned} H^n(\pi^{-1}(N), \pi^{-1}(N)_0; R) &\to H^n(F,F_0;R) \\ w &\mapsto w_{|(F,F_0)} \end{aligned}$$

induced by the inclusion $(F,F_0) \hookrightarrow (\pi^{-1}(N), \pi^{-1}(N)_0)$ equals the chosen generator u_F. We say that ξ is an R-oriented vector bundle if an R-orientation has been fixed. As usual, $R = \mathbf{Z}$ is to be assumed if no explicit mention of R is made.

Remark 3.15. If ξ is an R-oriented vector bundle and $B_1 \subseteq B$ is a subset of its base space, then the R-orientation of ξ induces an R-orientation on the restriction bundle $\xi_{|B_1}$.

Remark 3.16. If a vector bundle is oriented, then it is R-oriented for every commutative ring with unit R. This can be argued using the following general fact: if $A_\bullet$ is a chain complex of free $\mathbf{Z}$-modules, then there are well-defined maps

$$H^n(A;\mathbf{Z}) \to H^n(A;R)$$

that take a cohomology class $[\varphi] \in H^n(A;\mathbf{Z})$ represented by a cocycle $\varphi : A_n \to \mathbf{Z}$ to the cohomology class $[\varphi_R] \in H^n(A;R)$ represented by the cocycle

$$\begin{aligned} \varphi_R : A_n \otimes R &\to R \\ a \otimes r &\mapsto r \cdot \phi(\varphi(a)) \end{aligned}$$

where ϕ is the unique ring homomorphism $\mathbf{Z} \to R$.

Using this fact, one can check that the cohomology classes $u_F \in H^n(F,F_0;\mathbf{Z})$ are sent to generators of $H^n(F,F_0;R)$, also denoted u_F, and that the local compatibility condition is still satisfied.

3.2 The Euler class

The aim of this section is to give a proof of the following theorem and to discuss some of its consequences.

Theorem 3.17. *Let ξ be an oriented $\mathbf{R}^n$-bundle over B with total space E and let R be an commutative ring with unit. Then, ξ is canonically R-oriented (cf. Remark 3.16) and* *(i)* $H^i(E, E_0; R) = 0$ *for* $i < n$,

(ii) *there is a unique cohomology class* $u \in H^n(E, E_0; R)$ *whose restriction* $u_{|(F,F_0)}$ *equals the R-orientation choice* $u_F \in H^n(F, F_0; R)$ *for every fiber F, and*

(iii) *for every integer k, the map*

$$\begin{aligned} H^k(E; R) &\to H^{n+k}(E, E_0; R) \\ y &\mapsto u \smile y \end{aligned}$$

is an isomorphism.

Let us first talk about the consequences of this theorem. A quick one is the following.

Corollary 3.18. *(Thom isomorphism) For every integer k, the map*

$$\begin{aligned} \phi : H^k(B; R) &\to H^{n+k}(E, E_0; R) \\ x &\mapsto u \smile H^k(\pi)(x) \end{aligned}$$

is an isomorphism that will be called the Thom isomorphism.

Proof. Notice that π is a homotopy equivalence between E and B with inverse the zero section, so ϕ is just the composition of the isomorphism in cohomology induced by π and the isomorphism of Theorem 3.17. □

Theorem 3.17 also allows us to define the Euler class.

Definition 3.19. Let ξ be an oriented $\mathbf{R}^n$-bundle over B with total space E. The *Euler class* of ξ, denoted $e(\xi) \in H^n(B; \mathbf{Z})$, is the image of $u \in H^n(E, E_0; \mathbf{Z})$ under

$$H^n(E, E_0; \mathbf{Z}) \longrightarrow H^n(E; \mathbf{Z}) \xrightarrow{H^n(\pi)^{-1}} H^n(B; \mathbf{Z})$$

where the left homomorphism is induced by the inclusion $(E, \emptyset) \hookrightarrow (E, E_0)$. The Euler class of a smooth oriented manifold M is defined as the Euler class of its tangent bundle and is denoted by $e(M)$.

The next result is easily proven and, as we will see, it is satisfied for all characteristic classes.

Proposition 3.20. *(Naturality) Let* ξ, η *be oriented* $\mathbf{R}^n$*-bundles. If* $g : B(\xi) \to B(\eta)$ *is covered by an orientation preserving bundle map* $\xi \to \eta$*, then* $e(\xi) = H^n(g)(e(\eta))$.

Other important results that are somewhat satisfied for all characteristic classes are the following.

Proposition 3.21. *The Euler class of a trivial* $\mathbf{R}^n$*-bundle is zero.*

Proof. Use the above naturality property with ξ such a trivial bundle and η the trivial bundle with a point as base space. Since the nth cohomology of a point vanishes, the result follows. □

Proposition 3.22. *(Whitney product formula) Let* ξ *and* η *be oriented vector bundles over B. Then*

$$e(\xi \oplus \eta) = e(\xi) \smile e(\eta).$$

Here we regard $\xi \oplus \eta$ *as an oriented vector bundle with orientation on* $F_b(\xi) \oplus F_b(\eta)$ *given by an oriented basis for* $F_b(\xi)$ *followed by an oriented basis for* $F_b(\eta)$.

Proof. Let n, m be the fiber dimensions of ξ, η, respectively. Using the last diagram of Remark 1.57, it is easily seen that $u(\xi \times \eta) = u(\xi) \times u(\eta)$. Again by Remark 1.57, it follows that $e(\xi \times \eta) = e(\xi) \times e(\eta)$. Pulling back by the diagonal embedding $d : B \to B \times B$ and using Proposition 3.20 gives the result. □

Now we carry on by giving a proof of Theorem 3.17. We begin with a bunch of lemmas. Unless stated otherwise, coefficients in an arbitrary commutative ring with unit are to be considered.

Let $e \in H^1(\mathbf{R}, \mathbf{R}_0)$ be the cohomology class corresponding to $1 \in H^0(\mathbf{R}_+)$ under the sequence of isomorphisms

$$H^0(\mathbf{R}_+) \xleftarrow{\sim} H^0(\mathbf{R}_0, \mathbf{R}_-) \xrightarrow{\sim} H^1(\mathbf{R}, \mathbf{R}_0)$$

where the left one is induced by the inclusion $(\mathbf{R}_+, \emptyset) \hookrightarrow (\mathbf{R}_0, \mathbf{R}_-)$ (and it is an isomorphism by excision) and the right one is the connecting homomorphism of the long exact sequence of the triple $(\mathbf{R}, \mathbf{R}_0, \mathbf{R}_-)$ (and it is an isomorphism because $H^i(\mathbf{R}, \mathbf{R}_-) = 0$ since $\mathbf{R}_-$ is a deformation retract of $\mathbf{R}$). We denote the n-fold cross product $e \times \ldots \times e$ by $e^n \in H^n(\mathbf{R}^n, \mathbf{R}_0^n)$.

Lemma 3.23. *Let X be a topological space and let $A \subseteq X$ be an open subset. Then, for every integer $i \geq 0$, the map*

$$\begin{aligned} H^i(X, A) &\to H^{n+i}(\mathbf{R}^n \times X, \mathbf{R}^n \times A \cup \mathbf{R}_0^n \times X) \\ a &\mapsto e^n \times a \end{aligned}$$

is an isomorphism.

Proof. We proceed by induction on n. Assume we have already proven the initial case $n = 1$ and that the result is true for $n - 1$. Then, the correspondence $a \mapsto e^n \times a$ can be written as the composition of two isomorphisms $a \mapsto e^{n-1} \times a \mapsto e \times (e^{n-1} \times a) = e^n \times a$, where the equality follows from the associativity of the cross product. We can thus assume that $n = 1$ for the rest of the proof.

Step 1: Suppose that $A = \emptyset$. Let $a \in H^i(X)$. We claim that the diagram

$$\begin{tikzcd} H^0(\mathbf{R}_+) \arrow[d, "\times a"] & H^0(\mathbf{R}_0, \mathbf{R}_-) \arrow[l] \arrow[r] \arrow[d, "\times a"] & H^1(\mathbf{R}, \mathbf{R}_0) \arrow[d, "\times a"] \\ H^i(\mathbf{R}_+ \times X) & H^i(\mathbf{R}_0 \times X, \mathbf{R}_- \times X) \arrow[l] \arrow[r] & H^{i+1}(\mathbf{R} \times X, \mathbf{R}_0 \times X) \end{tikzcd}$$

commutes. Horizontal left arrows are excision isomorphisms and horizontal right arrows are the connecting homomorphisms of the long exact sequences of triples $(\mathbf{R}, \mathbf{R}_0, \mathbf{R}_-)$ and $(\mathbf{R} \times X, \mathbf{R}_0 \times X, \mathbf{R}_- \times X)$. The upper one has already been seen to be an isomorphism, whereas the lower one is also an isomorphism by an analogous reason. Commutativity of the left square is justified by Remark 1.57. Commutativity of the right square is argued by Remark 1.57 and also Remark 1.17. Now, we have $H^i(X) \cong H^i(\mathbf{R}_+ \times X)$ naturally by the correspondence $a \mapsto 1 \times a$. Finally, following the diagram around, we see that $e \times a \in H^{i+1}(\mathbf{R} \times X, \mathbf{R}_0 \times X)$ is the image of $a \in H^i(X)$ under a sequence of isomorphisms.

Step 2: Suppose that A is an arbitrary open subset of X. Let $z \in Z^1(\mathbf{R}, \mathbf{R}_0)$ be a cocycle representing $e \in H^1(\mathbf{R}, \mathbf{R}_0)$. Then, the diagram

$$\begin{tikzcd} 0 \arrow[r] & C^i(X, A) \arrow[r] \arrow[d, "z\times"] & C^i(X) \arrow[r] \arrow[d, "z\times"] & C^i(A) \arrow[r] \arrow[d, "z\times"] & 0 \\ 0 \arrow[r] & C^{i+1}(\mathbf{R}^n \times X, \mathbf{R}^n \times A \cup \mathbf{R}_0^n \times X) \arrow[r] & C^{i+1}(\mathbf{R} \times X, \mathbf{R}_0 \times X) \arrow[r] & C^{i+1}(\mathbf{R} \times A, \mathbf{R}_0 \times A) \arrow[r] & 0 \end{tikzcd}$$

commutes. Indeed, this follows again from Remark 1.57. The upper row is already known to be exact and the lower one can be easily seen to be. Now, by Remark 1.17, we have a commutative diagram in cohomology

$$\begin{tikzcd} \cdots \arrow[r] & H^i(X, A) \arrow[r] \arrow[d, "e\times"] & H^i(X) \arrow[r] \arrow[d, "e\times"] & H^i(A) \arrow[r] \arrow[d, "e\times"] & \cdots \\ \cdots \arrow[r] & H^{i+1}(\mathbf{R}^n \times X, \mathbf{R}^n \times A \cup \mathbf{R}_0^n \times X) \arrow[r] & H^{i+1}(\mathbf{R} \times X, \mathbf{R}_0 \times X) \arrow[r] & H^{i+1}(\mathbf{R} \times A, \mathbf{R}_0 \times A) \arrow[r] & \cdots \end{tikzcd}$$

in which both rows are exact. Notice that each $H^i(X,A) \to H^{i+1}(X\times\mathbf{R}, X\times\mathbf{R}_0 \cup A\times\mathbf{R})$ is surrounded by four maps that, by step 1, are isomorphisms. Hence, the Five Lemma [1] finishes the proof. $\square$

Lemma 3.24. *Let $f : (A_\bullet, \partial_\bullet^A) \to (B_\bullet, \partial_\bullet^B)$ be a linear map between free chain complexes over $\mathbf{Z}$ that satisfies $(\partial^B \circ f) = s(f\circ\partial^A)$ for some fixed $s = \{-1,+1\}$.*[2] *If f induces isomorphisms in cohomology*

$$H^n(f) : H^n(B;L) \to H^n(A;L)$$

for every integer n and every field L, then it induces isomorphisms in homology

$$H_n(f) : H_n(A;R) \to H_n(B;R)$$

and in cohomology

$$H^n(f) : H^n(B;R) \to H^n(A;R)$$

for every integer n and every commutative ring with unit R.

Proof. Denote the boundary maps of the chain complexes $A_\bullet$ and $B_\bullet$ by ∂^A and ∂^B, respectively. Let $C_n^f = A_{n-1}\oplus B_n$ and

$$\begin{aligned} \partial_n^f : C_n^f &\longrightarrow C_{n-1}^f \\ (a,b) &\longmapsto (-s\partial^A a, f(a) + \partial^B b). \end{aligned}$$

Since $\partial_{n-1}^f \circ \partial_n^f = 0$, $(C_\bullet^f, \partial_\bullet^f)$ is a free chain complex over $\mathbf{Z}$, which will be called the *mapping cone* of f. One can easily check that the sequence of chain complexes

$$0 \longrightarrow B_n \longrightarrow C_n^f \longrightarrow A_{n-1} \longrightarrow 0,$$

where the maps are the obvious ones, is exact. It is not hard to see that the connecting homomorphism of the corresponding long exact sequence coincides with $H_n(f) : H_n(A;R) \to H_n(B;R)$. Hence, $H_n(f)$ being an isomorphism for every integer n is equivalent to $H_n(C^f;R)$ being zero for every integer n.

Similarly, since the previous short exact sequence splits (notice that A_{n-1} is free), its dual is also exact and we obtain a long exact sequence in cohomology whose connecting homomorphism coincides with $H^n(f) : H^n(B;R) \to H^n(A;R)$. Hence, $H^n(f)$ being an isomorphism for every integer n is equivalent to $H^n(C^f;R)$ being 0 for every integer n. By hypothesis, the first statement is true for every field L so we must also have $H^n(C^f;L) = 0$ for every integer n and every field L. By the universal coefficient theorem for cohomology 1.39, we have an isomorphism $H^n(C^f;L) \cong \mathrm{Hom}_L(H_n(C^f;L), L)$. Thus, the homology modules $H_n(C^f;L) = H_n(C^f\otimes L; L)$ also vanish for every field L.

As already said, it suffices to show that $H_n(C^f;R) = H^n(C^f;R) = 0$ for every n and every commutative ring with unit R. Let us deal first with the homology case with $R = \mathbf{Z}$. Since $H_n(C^f\otimes\mathbf{Q}) = 0$, every cycle $z \in Z_n(C^f)$ has a multiple that is a boundary, so $H_n(C^f)$ is a torsion group. By a simple inductive argument, it suffices to check that every element of prime order in $H_n(C^f)$ is actually zero. Let $z \in Z_n(C^f)$ such that pz is a boundary for some prime p. Then,

$$pz = \partial c \quad (*)$$

for some $c \in C_{n+1}^f$. In particular, c is a cycle mod p (i.e. applying $\mathbf{Z}\to\mathbf{Z}_p$ to every coefficient of c, we obtain an element of $Z_{n+1}(C^f\otimes\mathbf{Z}_p)$). Since $H_n(C^f\otimes\mathbf{Z}_p) = 0$, c is also a boundary mod p, so we can write

$$c = \partial c' + pc'' \in C_{n+1}^f$$

[1] See [Hat01] p.129 for the statement and a proof.

[2] This can be thought as a generalization of the concept of *chain map*, for which $s = +1$. In either case, f sends cycles to cycles and boundaries to boundaries, so f induces morphisms in homology and, similarly, in cohomology.

for some $c' \in C^f_{n+2}$ and some $c'' \in C^f_{n+1}$. Substituting this expression into (∗), we get

$$pz = \partial(\partial c' + pc'') = p\partial c''.$$

Hence, $z = \partial c''$ is a boundary. This shows that $H_n(C^f) = 0$.

Now, for a general R, the universal coefficient theorem for homology 1.20 can be used to obtain $H_n(C^f; R) = 0$. Finally, the universal coefficient theorem for cohomology can be used again to obtain $H^n(C^f; R) = 0$. □

Now we will state and prove a series of claims that will combine to yield a proof of Theorem 3.17 in the general case. Notice first that condition (i) is a consequence of (iii), so we will restrict ourselves to show only (ii) and (iii).

Claim 3.25. *Theorem 3.17 is true if ξ is a trivial vector bundle.*

Proof. We can identify the total space of ξ with $\mathbf{R}^n \times B$ (cf. 3.5). By naturality (cf. Remark 1.57), for every $b \in B$, we have a commutative diagram

$$\begin{array}{ccc}
H^0(B;R) & \xrightarrow{\cong} & H^n(\mathbf{R}^n \times B, \mathbf{R}^n_0 \times B; R) \\
\downarrow & & \downarrow \\
H^0(N;R) & \xrightarrow{\cong} & H^n(\mathbf{R}^n \times N, \mathbf{R}^n_0 \times N; R) \\
\downarrow & & \downarrow \\
H^0(\{b\};R) & \xrightarrow{\cong} & H^n(\mathbf{R}^n \times \{b\}, \mathbf{R}^n_0 \times \{b\}; R)
\end{array}$$

where N is a neighborhood of b chosen by the local compatibility condition. The vertical arrows are induced by inclusions and the horizontal ones are isomorphisms by Lemma 3.23. The neighborhood N of every $b \in B$ is chosen by the local compatibility condition. Let $v_b \in H^0(\{b\}; R)$ be the cohomology class that maps to the preferred generator $u_{F_b} \in H^n(\mathbf{R}^n \times \{b\}, \mathbf{R}^n_0 \times \{b\}; R)$ under the bottom horizontal arrow of the diagram. A representative $\varphi_b \in C^0(\{b\}; R)$ of v_b can be thought of as a map $\{b\} \to R$, or equivalently, as an element $\varphi_b(b)$ of R. By the local compatibility condition and commutativity of the bottom square of the diagram, B can be covered by open subsets $N \subseteq B$ for which there are cohomology classes $v_N \in H^n(N; R)$ that restrict to $v_b \in H^n(\{b\}; R)$ for every $b \in N$. This means that if $\varphi_N \in C^0(N; R)$ ($\varphi_N : N \to R$) is a representative of the class v_N, then $\varphi_N(b) = \varphi_b(b)$ for every $b \in N$.

Now, let

$$\begin{aligned}\varphi : B &\to R \\ b &\mapsto \varphi_b(b).\end{aligned}$$

We want to show that φ is a cocycle. If $\sigma : [0, 1] \cong \Delta^1 \to B$ is a 1-simplex, its image being compact can be covered by finitely many open subsets of B, say $N_0, \dots, N_k$. We can then pick $0 = t_0 < t_1 < \dots < t_{l-1} < t_l = 1$ so that the image of each interval $[t_i, t_{i+1}]$ by σ lies in some N_j. Notice that the 1-chain $\sigma_{|[t_0,t_1]} + \dots + \sigma_{|[t_{l-1},t_l]}$ has the same boundary as σ, so we have

$$\varphi(\partial\sigma) = \sum_{i=0}^{l-1} \varphi(\partial\sigma_{|[t_i,t_{i+1}]}) = \sum_{i=0}^{l-1} \varphi_{N_{j_i}}(\partial\sigma_{|[t_i,t_{i+1}]}) = 0,$$

because each φ_{N_j} is already assumed to be a cocycle.

Hence, $\varphi \in C^0(B; R)$ represents a cohomology class $v \in H^0(B; R)$ that restricts to v_b for every $b \in B$. Using the commutativity of the diagram one last time, the image of v by the top horizontal arrow yields the desired cohomology class $u = e^n \times v \in H^n(\mathbf{R}^n \times B, \mathbf{R}^n_0 \times B; R)$. The uniqueness of this class is also clear from the uniqueness of the map $\varphi : B \to R$, so this proves part (ii) of the Theorem.

For part (iii), let $p_1 : (\mathbf{R}^n \times B, \mathbf{R}^n_0 \times B) \to (\mathbf{R}^n, \mathbf{R}^n_0)$, $p_2 : (\mathbf{R}^n \times B, \emptyset) \to (B, \emptyset)$ be the projection maps. Then, for every $y \in H^k(\mathbf{R}^n \times B; R)$, we have

$$\begin{aligned} u \smile y &= (e^n \times v) \smile y = (H^n(p_1)(e^n) \smile H^0(p_2)(v)) \smile y \\ &= H^n(p_1)(e^n) \smile (H^0(p_2)(v) \smile y) = H^n(p_1)(e^n) \smile H^k(p_2)(v \smile H^k(p_2)^{-1}(y)) \\ &= e^n \times (v \smile H^k(p_2)^{-1}(y)). \end{aligned}$$

The correspondence $y \mapsto H^k(p_2)^{-1}(y)$ is a well-defined isomorphism because p_2 is a homotopy equivalence. Since the cocycle $\varphi : B \to R$ takes each point $b \in B$ to a generator of R, the cohomology class v has an inverse with respect to the cup product. The correspondence $z \mapsto v \smile z$ is thus an isomorphism. The map $a \mapsto e^n \times a$ is also an isomorphism by Lemma 3.23. Hence, $u \smile y$ is the image of y by a sequence of isomorphisms, so $y \mapsto u \smile y$ is an isomorphism as well. This finishes the proof of the claim. □

Claim 3.26. *Assume that the base space B of ξ is the union of two open subsets B_1 and B_2. Denote $B_3 = B_1 \cap B_2$. If Theorem 3.17 is true for the restriction bundles $\xi_{|B_1}, \xi_{|B_2}$ and $\xi_{|B_3}$, then it is also true for ξ.*

Proof. Denote the total space of $\xi_{|B_i}$ by E^i. The relative Mayer-Vietoris sequence 1.49 for $X = E, Y = E_0$, $A = E^1, B = E^2, C = E^1_0, D = E^2_0$ is the long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H^i(E, E_0; R) \arrow[r, "f=(f_1,f_2)"] & H^i(E^1, E^1_0; R) \oplus H^i(E^2, E^2_0; R) \arrow[r, "g=g_1-g_2"] & H^i(E^3, E^3_0; R) \arrow[dll] \\ & H^{i+1}(E, E_0; R) \arrow[r] & \cdots \end{tikzcd}$$

where f_1, f_2, g_1 and g_2 are restriction homomorphisms. By hypothesis, there are unique cohomology classes $u_1 \in H^n(E^1, E^1_0; R), u_2 \in H^n(E^2, E^2_0; R), u_3 \in H^n(E^3, E^3_0; R)$ whose restrictions to each fiber F over B_1, B_2 or B_3 are the chosen generators $u_F \in H^n(F, F_0; R)$. By uniqueness of u_3, the classes u_1 and u_2 have image u_3 under the restriction homomorphism. Hence, $(u_1, u_2) \in \ker g = \operatorname{im} f$ and there is a $u \in H^n(E, E_0; R)$ whose restriction to (E^1, E^1_0), (E^2, E^2_0) is u_1 and u_2, respectively. In particular, u restricts to u_F for every fiber F over B. Now, if $\bar{u}$ is another class that satisfies the same property, then, by uniqueness of u_1 and u_2, it restricts to u_1 and u_2. But by hypothesis, we have $H^{n-1}(E^3, E^3_0; R) = 0$, so f is injective and hence $u = \bar{u}$, thus proving part (ii) of the Theorem.

For part (iii), let us first consider the Mayer-Vietoris sequence 1.48 for $X = E, A = E^1$ and $B = E^2$:

$$\begin{tikzcd} \cdots \arrow[r] & H^j(E; R) \arrow[r, "f=(f_1,f_2)"] & H^j(E^1; R) \oplus H^j(E^2; R) \arrow[r, "g=g_1-g_2"] & H^j(E^3; R) \arrow[dll] \\ & H^{j+1}(E; R) \arrow[r] & \cdots . \end{tikzcd}$$

Now, let φ, φ_i be cocycles representing the classes u, u_i, respectively. Then, the diagram

$$\begin{tikzcd} 0 \arrow[r] & C^k(E) \arrow[r] \arrow[d, "\varphi\smile"] & C^k(E^1) \oplus C^k(E^2) \arrow[r] \arrow[d, "(\varphi_1\smile, \varphi_2\smile)"] & C^k(E^3) \arrow[r] \arrow[d, "\varphi_3\smile"] & 0 \\ 0 \arrow[r] & C^{k+n}(E, E_0) \arrow[r] & C^{k+n}(E^1, E^1_0) \oplus C^{k+n}(E^2, E^2_0) \arrow[r] & C^{k+n}(E^3, E_0) \arrow[r] & 0 \end{tikzcd}$$

is easily seen to commute (coefficients in R are to be understood for the rest of the proof). Its rows are the short exact sequences that induce the previous Mayer-Vietoris sequences. Then, by naturality 1.17, we can attach both Mayer-Vietoris sequences together in a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & H^k(E) \arrow[r] \arrow[d, "u\smile"] & H^k(E^1) \oplus H^k(E^2) \arrow[r] \arrow[d, "(u_1\smile, u_2\smile)"] & H^k(E^3) \arrow[r] \arrow[d, "u_3\smile"] & H^{k+1}(E) \arrow[r] \arrow[d, "u\smile"] & \cdots \\ \cdots \arrow[r] & H^{k+n}(E,E_0) \arrow[r] & H^{k+n}(E^1,E_0^1) \oplus H^{k+n}(E^2,E_0^2) \arrow[r] & H^{k+n}(E^3,E_0^3) \arrow[r] & H^{k+n+1}(E,E_0) \arrow[r] & \cdots . \end{tikzcd}$$

Notice that each homomorphism $H^k(E) \xrightarrow{u\smile} H^{k+n}(E,E_0)$ is surrounded by four maps that, by hypothesis, are isomorphisms, so the Five Lemma finishes the proof. □

Claim 3.27. *Theorem 3.17 is true if the base space B is compact.*

Proof. Cover B with open subsets N that are the domains of local coordinate systems (N,h). Then, B is the union of a finite number of such domains, say $B = N_1 \cup \ldots \cup N_m$. We will prove by induction that the Theorem is true for the restriction bundles $\xi_{|N_1\cup\ldots\cup N_i}$ with $i = 1,\ldots,m$. The case $i = 1$ follows from Claim 3.25 because $\xi_{|N_1}$ is trivial. Assume that $\xi_{|N_1\cup\ldots\cup N_{i-1}}$ satisfies the properties (i)-(iii) of the Theorem. Again by Claim 3.25, these are also satisfied for $\xi_{|N_i}$ and $\xi_{|(N_1\cup\ldots\cup N_{i-1})\cap N_i}$ because they are trivial bundles. By Claim 3.26, the Theorem is thus true for $\xi_{|N_1\cup\ldots\cup N_i}$, as wanted. In particular, setting $i = m$ finishes the proof of the claim. □

For the next claim, we will need some definitions and results that are important by themselves.

Definition 3.28. A *directed set* is a pair $(I,\leq)$ where I is a non-empty set and $\leq$ is a preorder (reflexive and transitive relation) on I such that for every $i,j \in I$ there exists $k \in I$ with $i \leq k$ and $j \leq k$.

Definition 3.29. Let $(I,\leq)$ be a directed set, R an commutative ring with unit and $\{A_i\}_{i\in I}$ a family of R-modules indexed by I such that for every $i \leq j$ there is an R-linear map $f_{ij} : A_i \longrightarrow A_j$ satisfying

(i) $f_{ii} = \mathrm{id}_{A_i} : A_i \longrightarrow A_i$ and

(ii) $f_{ik} = f_{jk} \circ f_{ij}$ for every $i \leq j \leq k$.

Then, the pair $(\{A_i\},\{f_{ij}\})$ is called a *direct system* of R-modules over I. We define an equivalence relation $\sim$ on the disjoint union $\bigsqcup_{i\in I} A_i$ by

$$a_i \sim a_j \text{ with } a_i \in A_i, a_j \in A_j \Leftrightarrow f_{ik}(a_i) = f_{jk}(a_j) \text{ for some } k \in I \text{ with } i \leq k, j \leq k.$$

For every $a,b \in \bigsqcup_{i\in I} A_i$, we can then take $a',b' \in \bigsqcup_{i\in I} A_i$ lying in the same module A_k and satisfying $a \sim a'$, $b \sim b'$. We define a binary operation on $\bigsqcup_{i\in I} A_i / \sim$ by $[a] + [b] = [a' + b']$. We also define an action $r \cdot [a] = [ra]$ for every $r \in R$, $a \in \bigsqcup_i A_i$. It can be seen that these operations are well-defined and define an R-module structure on $\bigsqcup_{i\in I} A_i / \sim$. This R-module will be called the *direct limit* of the directed system $(\{A_i\},\{f_{ij}\})$ and will be denoted $\varinjlim A_i$.

Definition 3.30. Let $(I,\leq)$ be a directed set, R be an commutative ring with unit and $\{A_i\}_{i\in I}$ be a family of R-modules indexed by I such that for every $i \leq j$ there is an R-linear map $f_{ij} : A_j \longrightarrow A_i$ satisfying

(i) $f_{ii} = \mathrm{id}_{A_i} : A_i \longrightarrow A_i$ and

(ii) $f_{ik} = f_{ij} \circ f_{jk}$ for every $i \leq j \leq k$.

Then, the pair $(\{A_i\}, \{f_{ij}\})$ is called an *inverse system* of R-modules over I. The *inverse limit* of the pair $(\{A_i\}, \{f_{ij}\})$ is the R-submodule of $\prod_{i\in I} A_i$

$$\varprojlim A_i = \left\{ a = (a_i)_{i\in I} \in \prod_{i\in I} A_i : a_i = f_{ij}(a_j) \text{ for all } i \leq j \right\}$$

Proposition 3.31. *Let R be an commutative ring with unit, X and Y topological spaces, $A \subseteq X$. Let $f : X \to Y$ be a continuous map and let $\{X_i\}_{i\in I}$ be a covering of X whose index set I is a directed set with respect to the inclusion relation and satisfies that for every compact $C \subseteq Y$, $f^{-1}(C)$ is contained in some X_i. Set $A_i = A \cap X_i$. Then, the pairs $(\{H_m(X_i, A_i; R)\}_{i\in I}, \{f_{ij}\})$, $(\{H^m(X_i, A_i; R)\}_{i\in I}, \{g_{ij}\})$ form a directed system and an inverse system, respectively, with maps f_{ij}, g_{ij} induced by inclusion. Furthermore:*

(i) The natural map

$$\rho : \varinjlim H_m(X_i, A_i; R) \to H_m(X, A; R)$$

that takes a class $[\mu]$ with $\mu \in H_m(X_i, A_i; R)$ to $H_m(\gamma_i)(\mu) \in H_m(X, A; R)$ where γ_i is the inclusion $(X_i, A_i) \hookrightarrow (X, A)$ is a well-defined isomorphism.

(ii) If $H_{m-1}(X, A; R)$, $H_{m-1}(X_i, A_i; R)$ are zero or free R-modules, then the natural map

$$\kappa : H^m(X, A; R) \to \varprojlim H^m(X_i, A_i; R)$$

that takes $u \in H^m(X, A; R)$ to $(H^m(\gamma_i)(u))_{i\in I} \in \prod_{i\in I} H^m(X_i, A_i; R)$ is an isomorphism as well.

Proof. The fact that the given pairs form a directed system and an inverse system can be straightforwardly checked.

(i) If $\mu \in H_m(X_i, A_i; R)$ and $\eta \in H_m(X_j, A_j; R)$ represent the same class, then we can take $k \in I$ with $i \leq k, j \leq k$ and $f_{ik}(\mu) = f_{jk}(\eta)$. From the commutative diagram

$$\begin{tikzcd} H_m(X_j, A_j; R) \arrow[r, "f_{jk}"] \arrow[dr, bend right, "\gamma_j"'] & H_m(X_k, A_k; R) \arrow[d, "\gamma_k"] & H_m(X_i, A_i; R) \arrow[l, "f_{ik}"'] \arrow[dl, bend left, "\gamma_i"] \\ & H_m(X, A; R) & \end{tikzcd}$$

in which all arrows are induced by inclusion, we deduce that $H_m(\gamma_i)(\mu) = H_m(\gamma_j)(\eta)$ so the natural map is, indeed, well-defined.

Let $z \in Z_m(X, A; R)$ be a relative cycle and express it as a finite sum of m-simplices σ_α. Since Δ^m is compact, the set $C = \bigcup_\alpha (f \circ \sigma_\alpha)(\Delta^m)$ is compact in Y. Take X_i that contains $f^{-1}(C)$. Then X_i also contains $\bigcup_\alpha \sigma_\alpha(\Delta^m)$, so z is a relative cycle in $Z_m(X_i, A_i; R)$. This proves surjectivity of ρ.

Let $[\mu] \in \varinjlim H_m(X_i, A_i; R)$, $\mu \in H_m(X_i, A_i; R)$ such that $\rho_*([\mu]) = H_m(\gamma_i)(\mu) = 0$. This means that if $z \in Z_m(X_i, A_i; R)$ is a relative cycle representing μ, then z is a relative boundary in $B_m(X, A; R)$, so it can be written as $z = z' + \partial z''$ with $z' \in C_m(A; R)$ and $z'' \in C_{m+1}(X; R)$. Proceeding similarly as before, we can take X_j containing the simplices that form z' and z'' and then we take X_k containing X_i and X_j. Then, z is a relative boundary in $B_m(X_k, A_k; R)$, so it must be the zero element in $\varinjlim H_m(X_i, A_i; R)$. This proves injectivity of ρ.

(ii) The homomorphisms

$$h : H^m(X, A; R) \to \mathrm{Hom}_R(H_m(X, A; R), R),$$

$$h_i : H^m(X_i, A_i; R) \to \mathrm{Hom}_R(H_m(X_i, A_i; R), R)$$

of the universal coefficient theorem for cohomology are isomorphisms if $H_{m-1}(X,A;R)$, $H_{m-1}(X_i,A_i;R)$ are zero or free R-modules. Consider also the map

$$\chi : \mathrm{Hom}_R(\varinjlim H_m(X_i,A_i;R),R) \to \varprojlim \mathrm{Hom}_R(H_m(X_i,A_i;R),R)$$

that takes an arrow $\varinjlim H_m(X_i,A_i;R) \xrightarrow{\Psi} R$ to $(H_m(X_i,A_i;R) \xrightarrow{\Theta_i} R)_{i\in I}$, where $\Theta(a_i) = \Psi([a_i])$ for every $a_i \in H_m(X_i,A_i;R)$. This is easily seen to be an isomorphism. It is straightforward to verify that the diagram

$$\begin{tikzcd}
H^m(X,A;R) \arrow[rr, "\kappa"] \arrow[d, "h"] & & \varprojlim H^m(X_i,A_i;R) \arrow[d, "\prod_{i\in I} h_i"] \\
\mathrm{Hom}_R(H_m(X,A;R),R) \arrow[dr, "\rho^*"] & & \varprojlim \mathrm{Hom}_R(H_m(X_i,A_i;R),R) \\
& \mathrm{Hom}_R(\varinjlim H_m(X_i,A_i;R),R) \arrow[ur, "\chi"] &
\end{tikzcd}$$

commutes. Since every arrow apart from κ is an isomorphism, κ must be an isomorphism too.

□

As a consequence, we have the following corollary.

Corollary 3.32. *Let R be a field and let ξ be an $\mathbf{R}^n$-bundle over B with total space E and projection $\pi : E \to B$. Then, the natural R-linear maps*

$$\varinjlim H_m(\pi^{-1}(C);R) \to H_m(E;R), \qquad \varinjlim H_m(\pi^{-1}(C),\pi^{-1}(C)_0;R) \to H_m(E,E_0;R),$$
$$H^m(E;R) \to \varprojlim H^m(\pi^{-1}(C);R), \qquad H^m(E,E_0;R) \to \varprojlim H^m(\pi^{-1}(C),\pi^{-1}(C)_0;R),$$

where C varies over all compact subsets of B,[3] *are all isomorphisms.*

Claim 3.33. *Theorem 3.17 is true if R is a field.*

Proof. By Claim 3.27, for every compact $C \subseteq B$, there is a unique class $u_C \in H^n(\pi^{-1}(C),\pi^{-1}(C)_0;R)$ whose restriction to each fiber F gives the chosen generator. Now, take $u \in H^n(E,E_0;R)$ to be the unique class that has image $(u_C)_{C \text{ compact}}$ under the fourth isomorphism of Corollary 3.32. This class is now easily seen to be the only one that satisfies property (ii) of the Theorem.

By Proposition 1.53, we have commutative diagrams

$$\begin{tikzcd}
H^k(E;R) \arrow[r, "u\smile"] \arrow[d] & H^{k+n}(E,E_0;R) \arrow[d] \\
H^k(\pi^{-1}(C);R) \arrow[r, "u_C\smile"] & H^{k+n}(\pi^{-1}(C),\pi^{-1}(C)_0;R)
\end{tikzcd}$$

for each compact $C \subseteq B$. Taking the inverse limit over all compact subsets $C \subseteq B$ we get the commutative diagram

$$\begin{tikzcd}
H^k(E;R) \arrow[r, "u\smile"] \arrow[d, "\cong"'] & H^{k+n}(E,E_0;R) \arrow[d, "\cong"] \\
\varprojlim H^k(\pi^{-1}(C);R) \arrow[r, "\prod_C(u_C\smile)"] & \varprojlim H^{k+n}(\pi^{-1}(C),\pi^{-1}(C)_0;R)
\end{tikzcd}$$

in which the vertical arrows are isomorphisms by Corollary 3.32 and the bottom horizontal arrow is an isomorphism by Claim 3.27. Therefore, the top arrow is an isomorphism as well. This finishes the proof of the claim. □

[3] the inclusion relation gives a directed set because the union of two compact subsets is again compact.

Claim 3.34. *If the existence condition of part (ii) of Theorem 3.17 is true for $R = \mathbf{Z}$, then the whole theorem is true for any commutative ring with unit R. Furthermore, the maps*

$$\begin{aligned} H_{k+n}(E, E_0; R) &\to H_k(E; R) \\ \eta &\mapsto \eta \frown u \end{aligned}$$

are isomorphisms for every integer k.

Proof. One can check that the diagrams

$$\begin{tikzcd} C^n(E, E_0; \mathbf{Z}) \arrow[r] \arrow[d] & C^n(E, E_0; R) \arrow[d] \\ C^n(F, F_0; \mathbf{Z}) \arrow[r] & C^n(F, F_0; R) \end{tikzcd}$$

commute for each fiber F. Here, the vertical arrows are restrictions to fibers, the bottom horizontal map is the one in Remark 3.16 and the top one is a similar one with the pair (E, E_0) instead of (F, F_0). Now take a cocycle $\varphi \in C^n(E, E_0; \mathbf{Z})$ representing the class $u \in H^n(E, E_0; \mathbf{Z})$. Let φ_R be the image of φ under the top arrow. Then, the commutativity of the diagrams makes sure that $u^R = [\varphi_R] \in H^n(E, E_0; R)$ restricts to the desired R-orientation on each fiber. This shows the existence part of (ii) for arbitrary R.

Notice that the map

$$\begin{aligned} \frown \varphi : C_{k+n}(E, E_0; \mathbf{Z}) &\to C_k(E; \mathbf{Z}) \\ c &\mapsto c \frown \varphi \end{aligned}$$

is linear and satisfies $\partial(c \frown \varphi) = (-1)^n(\partial c \frown \varphi)$ (cf. Lemma 1.59). One can check that passing to chains with coefficients in R, the map $\frown \varphi$ becomes $\frown \varphi_R$ (in other words, $\frown \varphi \otimes \mathrm{id}_R = \frown \varphi_R$). Also, by Remark 1.61 the dual of $\frown \varphi_R$ is

$$\begin{aligned} \varphi_R \smile : C^k(E; R) &\to C^{k+n}(E, E_0; R) \\ \psi &\mapsto \varphi_R \smile \psi. \end{aligned}$$

Passing to homology and cohomology, we got maps

$$\frown u^R : H_{k+n}(E, E_0; R) \to H_k(E; R), \qquad u^R \smile : H^k(E; R) \to H^{k+n}(E, E_0; R).$$

Using uniqueness of the classes u^R when R is a field (cf. Claim 3.33), we deduce that the map in cohomology coincides with the one in part (iii) of the Theorem, which was already seen to be an isomorphism for every field R (again by Claim 3.33). Now we can use Lemma 3.24 to obtain that both maps are actually isomorphisms for arbitrary R.

We still need to prove that the classes $u^R \in H^n(E, E_0; R)$ are unique. To do it, for every $b \in B$, consider the diagram

$$\begin{tikzcd} H^0(B; R) \arrow[r, "H^0(\pi)"] \arrow[d] & H^0(E; R) \arrow[r, "u^R \smile"] \arrow[d] & H^n(E, E_0; R) \arrow[d] \\ H^0(\{b\}; R) \arrow[r, "H^0(\pi_{|F_b})"] & H^0(F_b; R) \arrow[r, "u_F^R \smile"] & H^n(F_b, F_{b0}; R) \end{tikzcd}$$

in which the vertical arrows are restrictions. The left square commutes because the corresponding square at the level of topological spaces commutes. The right square commutes because of Proposition 1.53. Now, assume that there is another class $\bar{u}^R \in H^n(E, E_0; R)$ that also restricts to $u^R_{F_b} \in H^n(F_b, F_{b0}; R)$ for every $b \in B$. Recall that $H^0(\pi)$ is an isomorphism because $\pi : E \to B$ is a homotopy equivalence and that $u^R \smile$ was proven to be an isomorphism as well. Starting at the top right of the diagram with $\overline{u}^R$, we obtain

$$\begin{tikzcd} z \arrow[d, mapsto] & y \arrow[l, mapsto] & \overline{u}^R \arrow[l, mapsto] \arrow[d, mapsto] \\ z(b) \arrow[r, mapsto] & z(b) \arrow[r, mapsto] & u_F = z(b)u_F \end{tikzcd}$$

Since we must have $u_F = z(b)u_F$ for every $b \in B$, the 0-cocycle $z : B \to R$ must be the constant map $1 \in R$. Mapping this constant map under the top arrows gives $\overline{u}^R = u^R \smile 1 = u^R$, as wanted. □

With this, we can finally give a general proof of Theorem 3.17.

Proof. (of Theorem 3.17) By Claim 3.27, for every compact $C \subseteq B$, there is a class

$$u_C \in H^n(\pi^{-1}(C), \pi^{-1}(C)_0; \mathbf{Z})$$

that restricts to the chosen $\mathbf{Z}$-orientation on each fiber. By Claim 3.34,

$$H_{n-1}(\pi^{-1}(C), \pi^{-1}(C)_0; \mathbf{Z}) \cong H_{-1}(\pi^{-1}(C); \mathbf{Z}) = 0.$$

By Proposition 3.31(i), we have an isomorphism

$$H_{n-1}(E, E_0; \mathbf{Z}) \cong \varinjlim H_{n-1}(\pi^{-1}(C), \pi^{-1}(C)_0; \mathbf{Z}) = 0.$$

Using these facts together with Proposition 3.31(ii), we have an isomorphism

$$\begin{aligned} \kappa : H^n(E, E_0; \mathbf{Z}) &\to \varprojlim H^n(\pi^{-1}(C), \pi^{-1}(C)_0; \mathbf{Z}) \\ w &\mapsto (w|_{(\pi^{-1}(C), \pi^{-1}(C)_0)})_C. \end{aligned}$$

Then, the class $u \in H^n(E, E_0; \mathbf{Z})$ whose image under κ is $(u_C)_C$ restricts to the chosen $\mathbf{Z}$-orientation and Claim 3.34 finishes the job. □

3.3 Chern classes

Here we introduce complex vector bundles and define Chern classes.

Definition 3.35. Let B be a topological space. A *complex vector bundle* ω over B consists of

(i) the given space B, which will be referred to as the *base space*,

(ii) a topological space $E = E(\omega)$ called the *total space*,

(iii) a continuous map $\pi : E \to B$ called the *projection map* and

(iv) a complex vector space structure on the sets $\pi^{-1}(b)$ for every $b \in B$.

Furthermore, the condition of local triviality must be satisfied. Namely, every $b \in B$ has a neighborhood $U \subseteq B$, an integer $n \geq 0$ and a homeomorphism

$$h : U \times \mathbf{R}^n \to \pi^{-1}(U)$$

so that for every $\overline{b} \in U$, the map

$$\begin{aligned} h_{\overline{b}} = h(\overline{b}, \bullet) : \mathbf{R}^n &\to \pi^{-1}(\overline{b}) \\ x &\mapsto h(\overline{b}, x) \end{aligned}$$

is a $\mathbf{C}$-isomorphism. Such a pair (U, h) is called *local coordinate system* for ω about b. The vector space $\pi^{-1}(b)$ is also denoted by $F_b(\omega)$ (or simply F_b) and is called *fiber* over b. If U can be chosen to be the entire base space B, then ω will be called a *trivial bundle*.

Remark 3.36. Because of the local triviality property, n is a locally constant function of b. For our purposes, though, n will always be constant and its value will be specified by saying that ω is a $\mathbf{C}^n$*-bundle* over B.

Definition 3.37. If ω is a $\mathbf{C}^n$-bundle over B, we can forget about the complex vector space structure to obtain an $\mathbf{R}^{2n}$-bundle over B, denoted $\omega_{\mathbf{R}}$.

This procedure of forgetting the complex structure yields a canonical preferred orientation on $\omega_{\mathbf{R}}$:

Proposition 3.38. *Let ω be a $\mathbf{C}^n$-bundle. Then, $\omega_{\mathbf{R}}$ is canonically oriented.*

Before giving a proof, recall that an orientation of an $\mathbf{R}^n$-bundle is a choice of generators $u_F \in H^n(F, F_0; \mathbf{Z})$ for each fiber F subject to a local compatibility condition (cf. 3.14). Equivalently, one can choose an orientation on each fiber given by an ordered basis and require B to be covered by local coordinate systems (U, h) such that $h(b, \bullet) : \mathbf{R}^n \to F_b$ is an orientation preserving isomorphism for every $b \in U$. This definition is more convenient to prove the proposition, so we will adopt it for a moment.

Proof. Let V be an n-dimensional complex vector space. We define an orientation of V as a $2n$-dimensional real vector space as follows. Start by taking a $\mathbf{C}$-basis $\{a_1, \dots, a_n\}$ for V. Then, $\{a_1, ia_1, \dots, a_n, ia_n\}$ is an ordered $\mathbf{R}$-basis for V and we can consider the orientation of V induced by it. We claim that this is well-defined. To prove it, assume we had taken another $\mathbf{C}$-basis $\{b_1, \dots, b_n\}$. Since the linear group $GL(n, \mathbf{C})$ is path-connected as a topological space,[4] there is a continuous path $M : [0, 1] \to GL(n, \mathbf{C})$ from the identity to the matrix whose columns are the components of $b_1, \dots, b_n$ with respect to the $\mathbf{C}$-basis $\{a_1, \dots, a_n\}$. For each $t \in [0, 1]$, let $c_1(t), \dots, c_n(t)$ be the columns of $M(t)$. When considered as vectors in the $\mathbf{C}$-basis $\{a_1, \dots, a_n\}$, these form a $\mathbf{C}$-basis $\{c_1(t), \dots, c_n(t)\}$ for V. For each $t \in [0, 1]$ we thus have an $\mathbf{R}$-basis $\{c_1(t), ic_1(t), \dots, c_n(t), ic_n(t)\}$. Let $M_{\mathbf{R}}(t)$ denote the matrix whose columns are the coordinates of $c_1(t), ic_1(t), \dots, c_n(t), ic_n(t)$ with respect to the $\mathbf{R}$-basis $\{a_1, ia_1, \dots, a_n, ia_n\}$. Since $M_{\mathbf{R}}(t)$ is continuous and $\det M_{\mathbf{R}}(0) = 1$, we must have $\det M_{\mathbf{R}}(1) > 0$, so the bases $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ induce the same orientation.

Taking such orientation on each fiber, the compatibility condition is easily checked and the result follows. □

Lemma 3.39. *(Gysin sequence) Let ξ be an $\mathbf{R}^n$-bundle with projection map $\pi : E \to B$ and denote $\pi_0 = \pi_{|E_0} : E_0 \to B$. Then, there is long exact sequence with integer coefficients*

$$\cdots \longrightarrow H^i(B) \xrightarrow{e(\xi)\smile} H^{i+n}(B) \xrightarrow{H^{i+n}(\pi_0)} H^{i+n}(E_0) \longrightarrow H^{i+1}(B) \longrightarrow \cdots$$

and will be referred to as Gysin sequence.

Proof. Consider the long exact sequence of the pair (E, E_0):

$$\cdots \longrightarrow H^j(E, E_0) \longrightarrow H^j(E) \longrightarrow H^j(E_0) \longrightarrow H^{j+1}(E, E_0) \longrightarrow \cdots .$$

Using the isomorphisms $H^{j-n}(E) \xrightarrow{u\smile} H^j(E, E_0)$ (cf. Theorem 3.17), we can substitute $H^j(E, E_0)$ by $H^{j-n}(E)$ to yield

$$\cdots \longrightarrow H^{j-n}(E) \xrightarrow{g} H^j(E) \longrightarrow H^j(E_0) \longrightarrow H^{j-n+1}(E) \longrightarrow \cdots ,$$

where $g(x) = (u \smile x)_{|E} = u_{|E} \smile x$ for every $x \in H^{j-n}(E)$. Finally, substituting $H^*(E)$ by $H^*(B)$ (recall that $\pi : E \to B$ is a homotopy equivalence), we obtain the desired sequence. □

[4] for any matrix $A \in GL(n, \mathbf{C})$, $\det(t \cdot \mathrm{id} + (1-t) \cdot A)$ has, at most, n complex roots in $[0, 1]$ which are avoidable by slightly changing the path.

Definition 3.40. Let ω be a $\mathbf{C}^n$-bundle $E \xrightarrow{\pi} B$. We define the $\mathbf{C}^{n-1}$-bundle ω_0 over E_0 as follows. Consider the $\mathbf{C}^n$-bundle $\pi_0^*\omega$ over E_0. Let η be the sub-bundle of $\pi_0^*\omega$ with total space

$$\{(u,v) \in E_0 \times E | \pi_0(u) = \pi(v), v \in \langle u \rangle\}.$$

Finally, take ω_0 to be the quotient of $\pi_0^*\omega$ by η.

Now we can finally give a definition of the Chern classes.

Definition 3.41. The *Chern classes* $c_i(\omega) \in H^{2i}(B;\mathbf{Z})$ are defined by induction on i as follows. We first take the top Chern class $c_n(\omega)$ to be the Euler class $e(\omega_\mathbf{R}) \in H^{2n}(B;\mathbf{Z})$. Notice that since $H^j(B;\mathbf{Z}) = 0$ for $j < 0$, every map

$$H^{2i}(\pi_0) : H^{2i}(B;\mathbf{Z}) \to H^{2i}(E_0;\mathbf{Z})$$

in the Gysin sequence 3.39 is an isomorphism. For $i < n$, we may then set $c_i(\omega) = H^{2i}(\pi_0)^{-1}(c_i(\omega_0))$. For $i > n$, we set $c_i(\omega) = 0$. The *total Chern class* is defined as

$$c(\omega) = 1 + c_1(\omega) + \dots + c_n(\omega) \in H^{\Pi}(B;\mathbf{Z}).$$

Furthermore, given a complex manifold M, we write $c(M)$ and $c_i(M)$ to denote the Chern classes of its tangent bundle equipped with the natural complex structure.

The following properties are analogous to Propositions 3.20-3.22.

Proposition 3.42. *(Naturality) Let ω, χ be $\mathbf{C}^n$-bundles. If $g : B(\omega) \to B(\chi)$ is covered by a bundle map $\omega \to \chi$ that is complex linear in each fiber, then $c_i(\omega) = H^{2i}(g)(c_i(\chi))$ for every integer i. The total Chern classes are related by $c(\omega) = g^*c(\chi)$, where we denote $g^* = H^*(g)$.*

Proposition 3.43. *Let ω be a $\mathbf{C}^n$-bundle and let ϵ^k be the trivial $\mathbf{C}^k$-bundle over $B(\omega)$. Then $c(\omega \oplus \epsilon^k) = c(\omega)$.*

Proposition 3.44. *(Whitney product formula) Let ω, χ be complex vector bundles over the same base space. Then $c(\omega \oplus \chi) = c(\omega)c(\chi)$.*

As an example, we now compute the total Chern class of the complex projective space $\mathbf{C}P^n$. The result is summarized in the following theorem.

Theorem 3.45. $c(\mathbf{C}P^n) = (1+a)^{n+1}$, *where a is a generator of $H^2(\mathbf{C}P^n;\mathbf{Z}) \cong \mathbf{Z}$.*

We will need some definitions and computations first.

Definition 3.46. Let ω be a $\mathbf{C}^n$-bundle. The conjugate bundle $\overline{\omega}$ is the $\mathbf{C}^n$-bundle with the same base space, total space and projection map as ω but with conjugate complex structure on the fibers. Namely, the action $\lambda \cdot e$ on a fiber $F(\overline{\omega})$ is the action $\overline{\lambda} \cdot e$ on a fiber $F(\omega)$.

The Chern classes of a complex vector bundle and those of its conjugate are related as follows.

Lemma 3.47. $c_i(\overline{\omega}) = (-1)^i c_i(\omega)$.

Proof. For each fiber F of ω, take a complex basis $v_1, \dots, v_n \in F$. Then, the real basis $v_1, iv_1, \dots, v_n, iv_n$ determines the preferred orientation of F as a fiber of $\omega_\mathbf{R}$. Similarly, the real basis $v_1, -iv_1, \dots, v_n, -iv_n$ determines the preferred orientation of F as a fiber of $\overline{\omega}_\mathbf{R}$. Hence, $\omega_\mathbf{R}$ and $\overline{\omega}_\mathbf{R}$ have the same orientation if n is even and the opposite one if n is odd, so

$$c_n(\overline{\omega}) = e(\overline{\omega}_\mathbf{R}) = (-1)^n e(\omega_\mathbf{R}) = (-1)^n c_n(\omega).$$

Now, assume the result is true for complex bundles of rank smaller than n. Then:

$$\begin{aligned} c_i(\overline{\omega}) &= H^{2i}(\pi_0)^{-1} c_i((\overline{\omega})_0) \overset{(*)}{=} H^{2i}(\pi_0)^{-1} c_i(\overline{\omega_0}) \overset{(**)}{=} H^{2i}(\pi_0)^{-1}((-1)^i c_i(\omega_0)) \\ &= (-1)^i H^{2i}(\pi_0)^{-1}(c_i(\omega_0)) = (-1)^i c_i(\omega). \end{aligned}$$

At $(*)$ we have used that $(\overline{\omega})_0 \cong \overline{\omega_0}$ under the identity, and $(**)$ holds by the induction hypothesis. ☐

Definition 3.48. The canonical complex line bundle $\gamma^1 = \gamma^1(\mathbf{C}^{n+1})$ is the $\mathbf{C}^1$-bundle over $\mathbf{C}P^n$ with total space

$$E(\gamma^1) = \{(\ell, v) \in \mathbf{C}P^n \times \mathbf{C}^{n+1} : v \in \ell\}$$

and obvious projection $(\ell, v) \mapsto \ell$.

As a first step towards the proof of Theorem 3.45, we will show the following.

Claim 3.49. $c(\mathbf{C}P^n) = (1 - c_1(\gamma^1))^{n+1}$

Proof. Let ϵ^{n+1} be the trivial $\mathbf{C}^{n+1}$-bundle over $\mathbf{C}P^n$. We equip it with the standard Hermitian metric on each fiber. Since γ^1 is a sub-bundle of ϵ^{n+1}, we can consider the orthogonal complement[5] ω^n of γ^1 in ϵ^{n+1}. There is a natural identification of $\tau(\mathbf{C}P^n)$, the tangent bundle of $\mathbf{C}P^n$, with $\mathrm{Hom}_{\mathbf{C}}(\gamma^1, \omega^n)$. Indeed, the projection $\mathbf{C}^{n+1} \setminus \{0\} \xrightarrow{p} \mathbf{C}P^n$ is a submersion with $\ker(d_z p) = \langle z \rangle$. Choosing $z \in \mathbf{C}^{n+1}$ such that $p(z) = \ell$, this allows us to identify $T_\ell(\mathbf{C}P^n)$ with $\mathbf{C}^{n+1}/\ell \cong \ell^\perp$ under the isomorphism $\varphi_z : \ell^\perp \to T_\ell(\mathbf{C}P^n)$ induced by $d_z p$. Of course, this identification depends on the choice of $z \in \mathbf{C}^{n+1}$, so it is certainly not natural yet. To drop such dependence, we identify each vector $\nu \in T_\ell(\mathbf{C}P^n)$ with the linear map

$$\begin{aligned} \ell &\to \ell^\perp \\ z &\mapsto (\varphi_z)^{-1}(\nu) \end{aligned}$$

Since every step of this identification can be locally written in terms of complex analytic functions, it defines a continuous function between the total spaces of $\tau(\mathbf{C}P^n)$ and $\mathrm{Hom}_{\mathbf{C}}(\gamma^1, \omega^n)$. As the restriction of this function to each fiber is an isomorphism, we have the desired isomorphism $\tau(\mathbf{C}P^n) \cong \mathrm{Hom}_{\mathbf{C}}(\gamma^1, \omega^n)$. Adding the trivial line bundle $\epsilon^1 \cong \mathrm{Hom}_{\mathbf{C}}(\gamma^1, \gamma^1)$ over $\mathbf{C}P^n$ to each side, we obtain:

$$\begin{aligned} \tau(\mathbf{C}P^n) \oplus \epsilon^1 &\cong \mathrm{Hom}_{\mathbf{C}}(\gamma^1, \omega^n) \oplus \mathrm{Hom}_{\mathbf{C}}(\gamma^1, \gamma^1) \cong \mathrm{Hom}_{\mathbf{C}}(\gamma^1, \omega^n \oplus \gamma^1) \\ &\cong \mathrm{Hom}_{\mathbf{C}}(\gamma^1, \epsilon^{n+1}) \cong \mathrm{Hom}_{\mathbf{C}}(\gamma^1, \epsilon^1 \oplus \overset{(n+1)}{\cdots} \oplus \epsilon^1) \\ &\cong \mathrm{Hom}_{\mathbf{C}}(\gamma^1, \epsilon^1) \oplus \overset{(n+1)}{\cdots} \oplus \mathrm{Hom}_{\mathbf{C}}(\gamma^1, \epsilon^1) \cong \bar{\gamma}^1 \oplus \overset{(n+1)}{\cdots} \oplus \bar{\gamma}^1. \end{aligned}$$

The last isomorphism above is given by taking each $v \in F(\bar{\gamma}^1)$ to $\langle \bullet, v \rangle \in \mathrm{Hom}_{\mathbf{C}}(F(\gamma^1), \mathbf{C})$, where $\langle \bullet, \bullet \rangle$ denotes the Hermitian metric on $\mathbf{C}^{n+1}$. To finish the proof of the claim, we use Propositions 3.43, 3.44 and Lemma 3.47.

$$c(\mathbf{C}P^n) = c(\tau(\mathbf{C}P^n) \oplus \epsilon^1) = c(\bar{\gamma}^1 \oplus \overset{(n+1)}{\cdots} \oplus \bar{\gamma}^1) = (1 - c_1(\gamma^1))^{n+1}.$$

□

Now we can finish the computation of the total Chern class of $\mathbf{C}P^n$.

Proof. (of Theorem 3.45) It suffices to show that $c_1(\gamma^1)$ is a generator of $H^2(\mathbf{C}P^n; \mathbf{Z})$. Lemma 3.39 gives the Gysin sequence associated to γ^1 (integer coefficients are understood):

$$\cdots \longrightarrow H^{i+1}(E_0) \longrightarrow H^i(\mathbf{C}P^n) \xrightarrow{c_1(\gamma^1)\smile} H^{i+2}(\mathbf{C}P^n) \longrightarrow H^{i+2}(E_0) \longrightarrow \cdots$$

Notice that E_0 can be homeomorphically identified with $\mathbf{C}^{n+1} \setminus \{0\}$, which is homotopically equivalent to $\mathbb{S}^{2n+1}$. The Gysin sequence thus reduces to

$$0 \longrightarrow H^i(\mathbf{C}P^n) \xrightarrow{c_1(\gamma^1)\smile} H^{i+2}(\mathbf{C}P^n) \longrightarrow 0$$

for every integer $0 \leq i \leq 2n - 2$. We thus have isomorphisms

$$H^0(\mathbf{C}P^n) \cong H^2(\mathbf{C}P^n) \cong \ldots \cong H^{2n}(\mathbf{C}P^n)$$

under the cup product by $c_1(\gamma^1)$. Since $\mathbf{C}P^n$ is path-connected, all of these groups are isomorphic to $\mathbf{Z}$ and the result follows. □

[5] Although this was defined for real vector bundles in 3.12, it analogously applies to complex ones.

Remark 3.50. The Gysin sequence also gives isomorphisms

$$0 \cong H^1(\mathbf{C}P^n) \cong H^3(\mathbf{C}P^n) \cong \ldots \cong H^{2n-1}(\mathbf{C}P^n).$$

In particular, we have also proved that the cohomology ring $H^*(\mathbf{C}P^n;\mathbf{Z})$ is generated by $c_1(\gamma^1) \in H^2(\mathbf{C}P^n;\mathbf{Z})$ and is isomorphic to $\mathbf{Z}[x]/(x^{n+1})$, where x is identified with $c_1(\gamma^1)$.

3.4 Pontrjagin classes

Before we state the definition of Pontrjagin classes, we give some preliminary results.

Definition 3.51. Given an n-dimensional real vector space V, we can consider its *complexification* $V \otimes_{\mathbf{R}} \mathbf{C}$, an n-dimensional complex vector space. Applying this to every fiber of an $\mathbf{R}^n$-bundle ξ, we obtain a canonical $\mathbf{C}^n$-bundle $\xi \otimes \mathbf{C}$ called the complexification of ξ.

Lemma 3.52. *For every $\mathbf{R}^n$-bundle ξ, the $\mathbf{C}^n$-bundles $\xi \otimes \mathbf{C}$ and $\overline{\xi \otimes \mathbf{C}}$ are isomorphic.*

Proof. The map $x + iy \mapsto x - iy$ is a homeomorphism from $E(\xi \otimes \mathbf{C})$ onto $E(\overline{\xi \otimes \mathbf{C}})$ and is $\mathbf{C}$-linear in each fiber. □

Definition 3.53. The *ith Pontrjagin class* $p_i(\xi) \in H^{4i}(B;\mathbf{Z})$ is defined to be $(-1)^i c_{2i}(\xi \otimes \mathbf{C})$. The *total Pontrjagin class* is defined as

$$p(\xi) = 1 + p_1(\xi) + p_2(\xi) + \ldots \in H^{\prod}(B;\mathbf{Z}).$$

For a smooth manifold M, we denote the Pontrjagin classes of its tangent bundle by $p_i(M)$ and $p(M)$.

Remark 3.54. Odd Chern classes are ignored as they have order 2. This is an immediate consequence of Lemmas 3.47 and 3.52.

Just as in the previous sections, Pontrjagin classes satisfy the following typical properties.

Proposition 3.55. *(Naturality) Let ξ, η be $\mathbf{R}^n$- bundles. If $g : B(\xi) \to B(\eta)$ is covered by a bundle map $\xi \to \eta$, then $p_i(\xi) = H^{4i}(g)(p_i(\eta))$ for every integer i. In terms of total Pontrjagin classes, $p(\xi) = g^* p(\eta)$.*

Proposition 3.56. *Let ξ be a $\mathbf{R}^n$-bundle and ϵ^k be the trivial $\mathbf{R}^k$-bundle over $B(\xi)$. Then $p(\xi \oplus \epsilon^k) = p(\xi)$.*

Proposition 3.57. *(Whitney product formula) Let ξ, η be vector bundles over the same base space. Then $2p(\xi \oplus \eta) = 2p(\xi)p(\eta)$.*

As an example, we now compute the total Pontrjagin class of the complex projective space $\mathbf{C}P^n$. As with Chern classes, we can summarize the result in the following theorem.

Theorem 3.58. $p(\mathbf{C}P^n) = (1 + a^2)^{n+1}$, *where a is any generator of $H^2(\mathbf{C}P^2, \mathbf{Z}) \cong \mathbf{Z}$.*

This will easily follow from two results below.

Lemma 3.59. *For any $\mathbf{C}^n$-bundle ω, there is an isomorphism $\omega_{\mathbf{R}} \otimes \mathbf{C} \cong \omega \oplus \overline{\omega}$.*

Proof. The endofunctors of the category of finite dimensional complex vector spaces $V \mapsto V_{\mathbf{R}} \otimes \mathbf{C}$ and $V \mapsto V \oplus \overline{V}$ are naturally isomorphic under the map

$$\begin{aligned} V_{\mathbf{R}} \otimes \mathbf{C} &\to V \oplus \overline{V} \\ v \otimes z &\mapsto (z \cdot v, z \cdot v) = (zv, \bar{z}v). \end{aligned}$$

Since naturally isomorphic functors induce isomorphic bundles, the claim follows. □

This has an immediate consequence on Pontrjagin classes.

Corollary 3.60. *Let ω be a $\mathbf{C}^n$-bundle. Denote $c_i = c_i(\omega)$ and $p_i = p_i(\omega_{\mathbf{R}})$. These are related by the formula*

$$\sum_{i=0}^{n}(-1)^i p_i = \left(\sum_{i=0}^{n}(-1)^i c_i\right)\left(\sum_{i=0}^{n} c_i\right).$$

We are now ready to finish the computation of $p(\mathbf{C}P^n)$.

Proof. (of Theorem 3.58) Denote $p_i = p_i(\mathbf{C}P^n) = p_i(\tau(\mathbf{C}P^n)_{\mathbf{R}})$ and $c_i = c_i(\mathbf{C}P^n)$. By the corollary and Theorem 3.45, we have

$$\sum_{i=0}^{n}(-1)^i p_i = (1-a)^{n+1}(1+a)^{n+1} = (1-a^2)^{n+1}$$

Thus, $p(\mathbf{C}P^n) = \sum_{i=0}^{n} p_i = (1+a^2)^{n+1}$. □

Another relevant computation that we will use later on is that of the Pontrjagin classes of the quaternionic projective space $\mathbf{H}P^n$. This was first done in [Hir53] and what follows is a slight remake of this work.

Theorem 3.61. $p(\mathbf{H}P^n) = (1+u)^{2n+2}(1+4u)^{-1}$, *where u is a generator of $H^4(\mathbf{H}P^n;\mathbf{Z}) \cong \mathbf{Z}$.*

Proof. Consider the smooth map

$$\begin{aligned} \mathbf{C}P^{2n+1} &\xrightarrow{\quad\pi\quad} \mathbf{H}P^n \\ [z_1 : z_2 : \dots : z_{2n+2}] &\longmapsto [z_1 + z_2 j : \dots : z_{2n+1} + z_{2n+2} j]. \end{aligned}$$

The fibers $\pi^{-1}(\pi[z_1 : \dots : z_{2n+2}])$ are the complex projective lines

$$\{[a(z_1, \dots, z_{2n+2}) + b(-\bar{z}_2, \bar{z}_1, \dots, -\bar{z}_{2n+2}, \bar{z}_{2n+1})] : a, b \in \mathbf{C}\} \cong \mathbf{C}P^1 \cong \mathbb{S}^2.$$

Fix a Riemannian metric on $\mathbf{C}P^{2n+1}$. By Remark 3.13, we can write the tangent bundle $\tau(\mathbf{C}P^{2n+1})$ as a Whitney sum $\tau_1 \oplus \tau_2$, where τ_1 and τ_2 are the sub-bundles of $\tau(\mathbf{C}P^{2n+1})$ consisting of vectors tangent and normal to the fiber $\mathbf{C}P^1$, respectively. τ_1 is actually independent of the choice of the Riemannian metric and inherits a complex vector bundle structure with complex fiber dimension 1. We want now to compute $p(\tau_1)$. To do this, it suffices to obtain $c(\tau_1)$ and use Corollary 3.60.

Fix a fiber ℓ of $\pi : \mathbf{C}P^{2n+1} \to \mathbf{H}P^n$. We may consider the restriction τ_ℓ of τ_1 to the fiber ℓ. Of course, τ_ℓ is just the tangent bundle $\tau(\ell)$. Since $\ell \cong \mathbf{C}P^1$, Theorem 3.45 and naturality imply $c_1(\tau_\ell) = 2a_\ell$, where $a_\ell \in H^2(\ell;\mathbf{Z})$ is a generator. Consider now the inclusion bundle map $\tau_\ell \to \tau_1$. Again by naturality, we have $c_1(\tau_\ell) = 2a_\ell = i^*(c_1(\tau_1))$, where $i : \mathbf{C}P^1 \hookrightarrow \mathbf{C}P^{2n+1}$ is the inclusion. But

$$i^* : H^2(\mathbf{C}P^{2n+1};\mathbf{Z}) \to H^2(\mathbf{C}P^1;\mathbf{Z})$$

is an isomorphism[6], so $c_1(\tau_1) = 2g$, where g is a generator of $H^2(\mathbf{C}P^{2n+1};\mathbf{Z})$. The total Chern class is then $c_1(\tau_1) = 1 + 2g$. By Corollary 3.60, we have computed $p(\tau_1) = 1 + 4g^2$.

Since $\pi : \mathbf{C}P^{2n+1} \to \mathbf{H}P^n$ is a fiber bundle, it is a submersion. By the regular value theorem, its differential $d\pi$ sends each vector of τ_1 to zero. Hence, $d\pi$ restricted to the fibers of τ_2 is an isomorphism, so it defines a bundle map $\tau_2 \to \tau(\mathbf{H}P^n)$. By naturality, $p(\tau_2) = \pi^* p(\mathbf{H}P^n)$. On the other hand, since all cohomology groups are free, Proposition 3.57 implies

$$p(\mathbf{C}P^n) = p(\tau_1)p(\tau_2).$$

[6] This can easily be seen regarding complex projective spaces as CW complexes. See, for instance, Lemma 2.34 in [Hat01].

Substituting what we have so far, gets us to

$$\pi^* p(\mathbf{H}P^n) = (1 + g^2)^{2n+2}(1 + 4g^2)^{-1}.$$

To finish the proof, it suffices to show that $\pi^* : H^{4i}(\mathbf{H}P^n; \mathbf{Z}) \to H^{4i}(\mathbf{C}P^{2n+1}; \mathbf{Z})$ are isomorphisms, so that we can set $\pi^* u = g^2$ and cancel out the π^*'s in the previous equation. As a first step towards this claim, it is a standard computation in algebraic topology that the cohomology groups $H^m(\mathbf{H}P^n; \mathbf{Z})$ are all zero except for $H^{4i}(\mathbf{H}P^n; \mathbf{Z}) \cong \mathbf{Z}$ for $0 \le i \le n$.[7]

Notice also that it only suffices to prove that $H^4(\pi) : H^4(\mathbf{H}P^n; \mathbf{Z}) \to H^4(\mathbf{C}P^{2n+1}; \mathbf{Z})$ is an isomorphism. Indeed, in this case we can already set $\pi^* u = g^2$ and by Proposition 1.53, each $u^i \in H^{4i}(\mathbf{H}P^n; \mathbf{Z}) \cong \mathbf{Z}$ is sent to the generator $g^{2i} \in H^{4i}(\mathbf{C}P^{2n+1}; \mathbf{Z}) \cong \mathbf{Z}$. It follows that the homomorphisms $H^{4i}(\pi)$ are all surjective, thus isomorphisms.

A further reduction we can do is to consider the commutative diagram

$$\begin{tikzcd}
H^4(\mathbf{H}P^n; \mathbf{Z}) \arrow[r, "\pi^*"] \arrow[d] & H^4(\mathbf{C}P^{2n+1}, \mathbf{Z}) \arrow[d] \\
H^4(\mathbf{H}P^1; \mathbf{Z}) \arrow[r, "(\pi_{|\mathbf{C}P^2})^*"] & H^4(\mathbf{C}P^2, \mathbf{Z}).
\end{tikzcd}$$

The vertical arrows are isomorphisms,[8] so it is enough to prove that the horizontal bottom arrow is an isomorphism as well. But this is now fairly simple in the cellular (co)homology setting, as $\pi_{|\mathbf{C}P^2}$ sends the 4-cell of $\mathbf{C}P^2$ to the 4-cell of $\mathbf{H}P^1$. This concludes the proof of the theorem. □

3.5 The Hirzebruch signature theorem

The aim of this section is to state and sketch a proof of the Hirzebruch signature theorem, as it is a fundamental ingredient in Milnor's argument of existence of exotic smooth spheres, which is presented in the next chapter. Unfortunately, the proof we sketch here heavily relies on a theorem of cobordism theory (namely, Theorem 3.63), which will be stated, but not proved.

The oriented cobordism ring Ω_*

The following material is mostly due to Thom in [Tho54], but it is taken from [MS74]. For an oriented smooth manifold M, we denote the same manifold with the opposite orientation by $-M$. We also use the symbol + to denote disjoint union.

Definition 3.62. Two smooth, compact, oriented m-dimensional manifolds M and N are *oriented cobordant* if there is a smooth compact oriented manifold with boundary B so that ∂B (with its induced orientation) is diffeomorphic to $M + (-N)$ under an orientation preserving diffeomorphism.

This can be shown to be an equivalence relation. Reflexive and symmetry properties are straightforward and the transitive one follows by taking collar neighborhoods (cf. Theorem 1.33) and gluing through the boundaries. We write Ω_m to denote the set of all oriented cobordism classes (i.e. equivalence classes under the previous relation). This is already an abelian group, but we further have well-defined associative bilinear products

$$\Omega_n \times \Omega_m \to \Omega_{n+m}$$

induced by taking the Cartesian product of manifolds. Hence, we have equipped $\Omega_* = \bigoplus_{i \ge 0} \Omega_i$ with a structure of graded ring.

[7] Again, the easiest way to check this is to regard $\mathbf{H}P^n$ as a CW complex and to use cellular (co)homology. See, for instance, page 140 in [Hat01].

[8] This follows again by Lemma 2.34 in [Hat01].

Theorem 3.63. *The tensor product $\Omega_* \otimes \mathbf{Q}$ is a polynomial algebra over $\mathbf{Q}$ with independent generators $\mathbf{C}P^2, \mathbf{C}P^4, \mathbf{C}P^6, \dots$.*

For a proof, one can see chapter 18 in [MS74] or [Sto68].

Pontrjagin numbers

Definition 3.64. Let M be a smooth, compact, oriented $4n$-manifold and let $I = i_1, \dots, i_r$ be a partition of n. The *Ith Pontrjagin number* is the integer

$$p_I[M] = p_{i_1} \cdots p_{i_r}[M] = \langle p_{i_1}(M) \cdots p_{i_r}(M), [M] \rangle,$$

where $[M] \in H_{4n}(M; \mathbf{Z})$ is the fundamental class of M with integer coefficients.

Pontrjagin numbers provide a necessary condition for a compact, oriented $4n$-manifold to be a boundary. Namely, we have the following.

Lemma 3.65. *For any smooth, compact, oriented $(4n+1)$-manifold with boundary B, all Pontrjagin numbers $p_I[\partial B]$ are zero.*

Proof. Write $M = \partial B$. As usual, denote the fundamental classes for B and M by $[B, M] \in H_{4n+1}(B, M)$ and $[M] \in H_{4n}(M)$, respectively (integer coefficients are to be understood throughout the proof). Let $\partial : H_{4n+1}(B, M) \to H_{4n}(M)$ and $\delta : H^{4n}(M) \to H^{4n+1}(B, M)$ be the connecting homomorphisms of the long exact sequences in homology and cohomology of the pair (B, M). Tracing back the definitions of these homomorphisms, it is easy to check that

$$\langle u, [M] \rangle = \langle \delta u, [B, M] \rangle.$$

Consider the tangent bundles $\tau(B)$ and $\tau(M)$. Taking a collar neighborhood of M in B allows us to write

$$\tau(B)_{|M} \cong \tau(M) \oplus \epsilon^1.$$

This and naturality imply $i^* p(B) = p(\tau(B)_{|M}) = p(M)$. The portion of the long exact sequence of the pair (B, M)

$$H^{4n}(B) \xrightarrow{i^*} H^{4n}(M) \xrightarrow{\delta} H^{4n+1}(B, M)$$

asserts that

$$\delta(p_{i_1}(M) \cdots p_{i_r}(M)) = (\delta i^*)(p_{i_1}(B) \cdots p_{i_r}(B)) = 0$$

for any partition $I = i_1, \dots, i_r$ of n. Hence,

$$p_I[M] = \langle p_{i_1}(M) \cdots p_{i_r}(M), [M] \rangle = \langle \delta(p_{i_1}(M) \cdots p_{i_r}(M)), [B, M] \rangle = 0.$$

□

As a corollary, we have:

Corollary 3.66. *Any partition $I = i_1, \dots, i_r$ of n provides a group homomorphism*

$$\begin{aligned} \Omega_{4n} &\to \mathbf{Z} \\ M &\mapsto p_I[M]. \end{aligned}$$

Proof. It suffices to prove $p_I[M + N] = p_I[M] + p_I[N]$ for arbitrary smooth, compact, oriented $4n$-manifolds M and N. But this is an easy consequence of the naturality of the Pontrjagin classes under the inclusions $M \hookrightarrow M + N$ and $N \hookrightarrow M + N$. □

Multiplicative sequences

Let $A_* = \bigoplus_{i\geq 0} A_i$ be a commutative graded $\mathbf{Q}$-algebra with unit. We denote the commutative ring consisting of formal sums $a_0 + a_1 + \ldots$ with $a_i \in A_i$ by $A^{\prod}$. Throughout this section, assign degree i to the indeterminate x_i and consider polynomials

$$K_1(x_1), K_2(x_1, x_2), \ldots, K_n(x_1, \ldots, x_n), \ldots$$

with coefficients in $\mathbf{Q}$ such that each K_n is homogeneous of degree n. For any $\mathbf{Q}$-algebra A_* as above, and any $a = 1 + a_1 + a_2 + \ldots \in A^{\prod}$, we denote $K(a) = 1 + K(a_1) + K_2(a_1, a_2) + \ldots + K_n(a_1, \ldots, a_n) + \ldots \in A^{\prod}$.

Definition 3.67. The sequence of polynomials $\{K_n\}$ is said to be a *multiplicative sequence* if for any $\mathbf{Q}$-algebra A_* as above, we have $K(ab) = K(a)K(b)$ for every $a, b \in A^{\prod}$ with leading term 1.

Multiplicative sequences are easily classified by the following result.

Lemma 3.68. *(Hirzebruch) Let $f(t) = 1 + \lambda_1 t + \lambda_2 t^2 + \ldots$ be a formal power series with coefficients in $\mathbf{Q}$. Then, there is one and only one multiplicative sequence $\{K_n\}$ satisfying $K(1 + t) = f(t)$ for the $\mathbf{Q}$-algebra $\mathbf{Q}[t]$. This sequence will be referred to as the multiplicative sequence belonging to the power series $f(t)$.*

For a proof of this lemma, see chapter 19 in [MS74].

Definition 3.69. Let M be a smooth, compact, oriented m-manifold. The *K-genus* $K[M]$ is defined to be zero if m is not divisible by 4 and to be

$$\langle K_n(p_1, \ldots, p_n), [M]\rangle \in \mathbf{Q}$$

if $m = 4n$, where $p_i = p_i(M)$.

Lemma 3.70. *Any multiplicative sequence $\{K_n\}$ gives rise to a ring homomorphism*

$$\begin{aligned} \Omega_* &\to \mathbf{Q} \\ M &\mapsto K[M]. \end{aligned}$$

Hence, it also gives rise to an algebra homomorphism $\Omega_* \otimes \mathbf{Q} \to \mathbf{Q}$.

Proof. Since $K[M]$ is a rational linear combination of Pontrjagin numbers, Corollary 3.66 makes sure that the map is well-defined and additive. Multiplicativity is proved by using the isomorphism

$$\tau(M_1 \times M_2) \cong \pi_1^* \tau(M_1) \oplus \pi_2^* \tau(M_2),$$

the Whitney product formula, naturality and the multiplicativity of $\{K_n\}$. □

The signature of a manifold

We now define the signature of a smooth, compact and oriented manifold, prove some properties about it and state the Hirzebruch signature theorem.

Definition 3.71. Let M be a smooth, compact, oriented m-manifold. The *signature* $\sigma(M)$ is defined to be zero if m is not divisible by 4 and to be as follows if $m = 4n$: regard the fundamental class $[M]$ as an element of the rational homology $H_{4n}(M; \mathbf{Q})$. Take a basis $a_1, \ldots, a_r \in H^{2n}(M; \mathbf{Q})$ so that the symmetric matrix

$$\left(\langle a_i \smile a_j, [M]\rangle\right)_{ij}$$

is diagonal.[9] The signature $\sigma(M)$ is then the number of positive entries minus the number of negative ones.

[9] This can be done, as the cup product pairing $H^{2n}(M; \mathbf{Q}) \times H^{2n}(M; \mathbf{Q}) \to \mathbf{Q}$ is symmetric bilinear and we are working over the field $\mathbf{Q}$.

Lemma 3.72. *The signature satisfies the following properties:*

(i) $\sigma(M) = 0$ *if* M *is the boundary of a compact, oriented manifold,*

(ii) $\sigma(M + N) = \sigma(M) + \sigma(N)$,

(iii) $\sigma(M \times N) = \sigma(M)\sigma(N)$.

As a consequence, the signature defines a ring homomorphism $\Omega_* \to \mathbf{Z}$ *and an algebra homomorphism* $\Omega_* \otimes \mathbf{Q} \to \mathbf{Q}$.

Property (ii) is straightforward. For properties (i) and (iii), see Theorem 8.2.1 in [Hir78].

Theorem 3.73. *(Hirzebruch signature theorem) Let* $\{L_n\}$ *be the multiplicative sequence belonging to the power series*

$$\frac{\sqrt{t}}{\tanh\sqrt{t}} = 1 + \frac{1}{3}t - \frac{1}{45}t^2 + \dots .$$

Then, the signature of any smooth, compact and oriented manifold is equal to its L-genus.

Proof. By lemmas 3.70, 3.72, and Theorem 3.63, it suffices to prove the equality $\sigma(M) = L[M]$ for $M = \mathbf{C}P^{2n}$. We start by computing the signature of $\mathbf{C}P^{2n}$. Let a be a generator of $H^2(\mathbf{C}P^{2n};\mathbf{Z})$. Recall that by proving Theorem 3.45, we also showed that $H^{2i}(\mathbf{C}P^{2n};\mathbf{Z})$ is generated by a^n for every $0 \le i \le 2n$. Thus, $\{a^n\}$ is a basis of $H^{2n}(\mathbf{C}P^{2n};\mathbf{Q})$ and the signature $\sigma(\mathbf{C}P^{2n})$ equals $\langle a^{2n}, [\mathbf{C}P^{2n}]\rangle$.

We now compute $L[\mathbf{C}P^{2n}]$. Recall that Theorem 3.58 gave us $p = p(\mathbf{C}P^{2n}) = (1+a^2)^{2n+1}$. It follows that

$$L(p) = L(1 + a^2)^{2n+1} = \left(\frac{a}{\tanh a}\right)^{2n+1}.$$

The L-genus $L[\mathbf{C}P^{2n}] = \langle L_n(p_1, \dots, p_n), [\mathbf{C}P^{2n}]\rangle$ is thus equal to $\langle a^{2n}, [\mathbf{C}P^{2n}]\rangle$ times the coefficient of z^{2n} in the power series of $\left(\frac{z}{\tanh z}\right)^{2n+1}$. To finish the proof, it then suffices to see that this coefficient is 1. This is purely a problem of complex analysis. Notice that the wanted coefficient is the residue of the function $\left(\frac{1}{\tanh z}\right)^{2n+1}$. We have

$$\frac{1}{2\pi i}\oint \frac{dz}{(\tanh z)^{2n+1}} = \frac{1}{2\pi i}\oint \frac{(1 + w^2 + w^4 + \dots)dw}{w^{2n+1}} = \frac{1}{2\pi i}\oint \frac{dw}{w} = 1,$$

where the first integral is through a small enough circle around the origin and in the first equality we have performed the change $w = \tanh z$, $dz = \frac{dw}{1-w^2} = 1 + w^2 + w^4 + \dots$. The second and third equalities are consequences of the residue theorem. $\square$

Chapter 4

Proof of the main theorem

This chapter replicates Milnor's transcendental paper *On Manifolds Homeomorphic to the 7-Sphere* ([Mil56]). The aim of what follows is then to prove the existence of exotic 7-spheres. Namely, we will show the following result.

Theorem 4.1. *For $k^2 \not\equiv 1$ (mod 7), the manifold M_k^7 constructed in Section 2.2 is homeomorphic to $\mathbb{S}^7$ but not diffeomorphic to $\mathbb{S}^7$.*

The *homeomorphic to* $\mathbb{S}^7$ part was already proven in the same Section 2.2 using Morse theory, so here we only deal with the *not diffeomorphic to* $\mathbb{S}^7$ part.

The general idea of the argument is fairly simple. In Section 4.1, we define a diffeomorphism invariant $\lambda(M)$ on certain smooth 7-manifolds M. In Section 4.2, we compute this invariant for the manifolds M_k^7 and see that it equals $k^2 - 1$ (mod 7). Since for the standard 7-sphere the invariant will be zero, this will conclude the proof.

4.1 The invariant $\lambda(M)$

Definition 4.2. Let M be a smooth, compact, oriented 7-manifold such that

(i) it is the boundary of a smooth, compact, oriented 8-manifold with boundary B,[1] and

(ii) $H^3(M) = H^4(M) = 0$.[2]

Write $\sigma(B)$ to denote the signature of the symmetric bilinear form

$$\begin{aligned} H^4(B, M; \mathbf{Q}) \times H^4(B, M; \mathbf{Q}) &\longrightarrow \mathbf{Q} \\ (a, b) &\longmapsto \langle a \smile b, [B, M] \rangle \end{aligned}$$

that is, the number of positive entries minus the number of negative ones when diagonalized (compare with Definition 3.71).

Because of the following portion of the long exact sequence of the pair (B, M)

$$0 \longrightarrow H^4(B, M) \xrightarrow{i^*} H^4(B) \longrightarrow 0,$$

the map i^* is an isomorphism. Let $p_1 = p_1(B) \in H^4(B)$ and

$$q(B) = \langle ((i^*)^{-1} p_1)^2, [B, M] \rangle.$$

The invariant $\lambda(M)$ is then defined to be $2q(B) - \sigma(B) \pmod 7$.

[1]Actually, this always holds, as the oriented cobordism group Ω_7 is trivial (see [Tho54]).

[2]Integer coefficients are to be understood throughout the whole chapter unless otherwise specified.

Of course, it is not clear that this definition is independent of the choice of the manifold with boundary B, so we should check that this is indeed the case.

Lemma 4.3. *$\lambda(M)$ is well-defined.*

Proof. Let B_1 and B_2 be smooth, compact, oriented 8-manifolds with boundary M. Consider the closed 8-manifold C obtained by gluing B_1 and B_2 along their common boundary M. This is easily proven to have an orientable smooth structure compatible with that of B_1 and B_2 by using of smooth collar neighborhoods (see Theorem 1.33). Take the fundamental class $[C] \in H_8(C)$ that corresponds to the pair $([B_1, M], -[B_2, M]) \in H_8(B_1, M) \oplus H_8(B_2, M)$ under the diagram

$$\begin{tikzcd} & & H_7(M) \\ 0 \arrow[r] & H_8(B_1,M) \oplus H_8(B_2,M) \arrow[r, "h_*"] & H_8(C,M) \arrow[u] \arrow[r] & 0 \\ & & H_8(C) \arrow[u, "j_*"] \\ & & 0 \arrow[u] \end{tikzcd}$$

where the horizontal and vertical sequences come from Mayer-Vietoris and the sequence of the pair (C, M), respectively. Write $q(C) = \langle p_1^2(C), [C]\rangle$. Since $L_2(x_1, x_2) = \frac{1}{45}(7x_2 - x_1^2)$, the Hirzebruch signature theorem gives

$$\sigma(C) = \left\langle \frac{1}{45}(7p_2(C) - p_1^2(C)), [C] \right\rangle \in \mathbf{Z}.$$

Taking integers mod 7, this yields

$$2q(C) - \sigma(C) \equiv 0 \pmod 7$$

To finish the proof, it suffices to show

$$\sigma(C) = \sigma(B_1) - \sigma(B_2), \tag{4.1}$$

$$q(C) = q(B_1) - q(B_2), \tag{4.2}$$

as combining the last three equations leads to $2q(B_1) - \sigma(B_1) \equiv 2q(B_2) - \sigma(B_2) \pmod 7$.

Consider the diagram

$$\begin{tikzcd} 0 & H^n(B_1,M) \oplus H^n(B_2,M) \arrow[l] \arrow[d, "i_1^* \oplus i_2^*"] & H^n(C,M) \arrow[l, "h^*"'] \arrow[d, "j^*"] & 0 \arrow[l] \\ 0 & H^n(B_1) \oplus H^n(B_2) \arrow[l] & H^n(C) \arrow[l, "s"'] & 0 \arrow[l] \end{tikzcd}$$

where, as before, the horizontal sequences come from Mayer-Vietoris and the vertical ones are portions of the long exact sequences of certain pairs. Notice that since $H^3(M) = H^4(M) = 0$, the vertical arrows are isomorphisms when $n = 4$. Thus, letting $a, b \in H^4(C)$, we may write $(a_1, a_2) = h^*(j^*)^{-1}a$ and $(b_1, b_2) = h^*(j^*)^{-1}b$. Then,

$$\begin{aligned} \langle a \smile b, [C]\rangle &= \left\langle j^*(h^*)^{-1}(a_1 \smile b_1, a_2 \smile b_2), [C]\right\rangle \\ &= \left\langle (a_1 \smile b_1, a_2 \smile b_2), (h_*)^{-1} j_*[C]\right\rangle \\ &= \langle (a_1 \smile b_1, a_2 \smile b_2), ([B_1, M], -[B_2, M])\rangle \\ &= \langle a_1 \smile b_1, [B_1, M]\rangle - \langle a_2 \smile b_2, [B_2, M]\rangle. \end{aligned} \tag{4.3}$$

This shows equation (4.1). Now, by naturality of the Pontrjagin classes, we have

$$sp_1(C) = (p_1(B_1), p_1(B_2)).$$

The commutativity of the last diagram, implies

$$j^*(h^*)^{-1}(i_1^* \oplus i_2^*)^{-1}(p_1(B_1), p_1(B_2)) = p_1(C).$$

But now, the computation of equation (4.3) with $a_1 = b_1 = (i_1^*)^{-1}p_1(B_1)$, $a_2 = b_2 = (i_2^*)^{-1}p_1(B_2)$ and $a = b = p_1(C)$ shows that

$$q(C) = \langle p_1^2(C), [C]\rangle = \langle ((i_1^*)^{-1}p_1(B_1))^2, [B_1, M]\rangle - \langle ((i_2^*)^{-1}p_1(B_2))^2, [B_2, M]\rangle = q(B_1) - q(B_2),$$

which shows (4.2) and finishes the proof. □

Remark 4.4. It should be pointed out in which sense is $\lambda(M)$ an invariant. Let M_1 and M_2 be two smooth, compact, oriented 7-manifolds. Suppose there is an orientation preserving diffeomorphism $\varphi : M_1 \to M_2$. Choose a compact, oriented 8-manifold B_1 with $\partial B_1 = M_1$. Then, using a collar neighborhood, we may take a compact oriented 8-manifold B_2 with $\partial B_2 = M_2$, and extend φ to an orientation preserving diffeomorphism $\phi : (B_1, M_1) \to (B_2, M_2)$. Then, it is straightforward to check that $\lambda(M_1) = \lambda(M_2)$, so λ is an invariant under orientation preserving diffeomorphisms.

Remark 4.5. Note that the λ invariant of the standard 7-sphere vanishes. Indeed, if $M = \mathbb{S}^7$, we can choose B to be the 8-disk D^8. Then, the cohomology group $H^4(D^8, \mathbb{S}^7)$ is trivial and hence $q(D^8) = \sigma(D^8) = 0$.

4.2 Computation of $\lambda(M_k^7)$

As in Lemma 2.11, we may consider the space B_k^8 obtained by taking two copies of $\mathbf{R}^4 \times D^4$ and gluing them through the diffeomorphism

$$g : (\mathbf{R}^4 \setminus \{0\}) \times D^4 \longrightarrow (\mathbf{R}^4 \setminus \{0\}) \times D^4$$
$$(u, v) \longmapsto (u', v') = \left(\frac{u}{\|u\|^2}, \frac{u^h v u^j}{\|u\|^{h+j}}\right).$$

Using a modification of Lemma 2.10, B_k^8 is seen to be a smooth, compact, orientable 8-manifold with boundary. Furthermore, $\partial B_k^8 = M_k^7$, so conditions (i) and (ii) of Definition 4.2 are satisfied for the manifolds M_k^7. We can thus talk about the invariant $\lambda(M_k^7)$.

We can also define orientable vector bundles ξ_{hj} over $\mathbb{S}^4$ for arbitrary integers h, j in an analogous manner. Namely, take as a total space E_{hj} two copies of $\mathbf{R}^4 \times \mathbf{R}^4$ and glue them with a diffeomorphism $g : (\mathbf{R}^4 \setminus \{0\}) \times \mathbf{R}^4 \longrightarrow (\mathbf{R}^4 \setminus \{0\}) \times \mathbf{R}^4$ defined as above. This is again a smooth manifold. Finally, take as projection $\pi_{hj} : E_{hj} \to \mathbb{S}^4$ the inverse of the stereographic projection by the north pole on the first component $\mathbf{R}^4$ of the first copy of $\mathbf{R}^4 \times \mathbf{R}^4$, and similarly by the south pole for the second copy of $\mathbf{R}^4 \times \mathbf{R}^4$.

Classification of fiber bundles over the sphere

In this subsection we intend to generalize the concept of vector bundles in the sense of allowing fibers to be arbitrary topological spaces. These new objects will be called *fiber bundles*. It turns out that fiber bundles over the sphere $\mathbb{S}^n$ are easily classified up to isomorphism by the homotopy class represented by a particular map, as is stated below in Theorem 4.12. This applies in particular to vector bundles, but for the sake of completeness and elegance, it is convenient to state everything in terms of fiber bundles and then restrict ourselves to the case we are interested in. The main reference is [Ste51], and an alternative one is [Hat17].

We begin by giving the definition of the notion of *coordinate bundle*.

Definition 4.6. A *coordinate bundle* ξ consists of the following objects:

(i) A topological space E called the *total space*,

(ii) a topological space B called the *base space*,

(iii) a continuous map $\pi : E \to B$ called the *projection*,

(iv) a topological space F called the *fiber*,

(v) a topological group G called the *structure group*,

(vi) a faithful and continuous action of G on F, and

(vii) a collection $\{(U_i, \phi_i)\}_{i\in I}$ called *local trivialization*, where $\{U_i\}_{i\in I}$ is an open cover of B and

$$\phi_i : U_i \times F \to \pi^{-1}(U_i)$$

are homeomorphisms. The objects U_i, ϕ_i and (U_i, ϕ_i) are called *coordinate neighborhood*, *coordinate function* and *coordinate system*, respectively.

These are required to satisfy the properties below:

(a) $(\pi \circ \phi_i)(b, x) = b$ for every $i \in I$, $b \in U_i$, $x \in F$.

(b) Let $\phi_{i,b}$ for $i \in I$ and $b \in U_i$ be the map

$$\begin{aligned} \phi_{i,b} : F &\to \pi^{-1}(b) \\ x &\mapsto \phi_i(b, x). \end{aligned}$$

For each $i, j \in I$ and each $b \in U_i \cap U_j$, the homeomorphism $\phi_{i,b}^{-1} \circ \phi_{j,b} : F \to F$ must coincide with the operation of a unique element of G, which we denote by $g_{ij}(b)$.

(c) For each $i, j \in I$, the map $g_{ij} : U_i \cap U_j \to G$ defined above must be continuous.

Definition 4.7. We say that two coordinate bundles ξ and η with local trivializations $\{(U_i, \phi_i)\}_{i\in I}$ and $\{(U_j, \phi_j)\}_{j\in J}$ ($I \cap J = \emptyset$), respectively, are equivalent if they have the same objects from (i) to (vi) in the definition above, and $\{(U_i, \phi_i)\}_{i\in I\cup J}$ is the local trivialization of a coordinate bundle also having these same objects.

It is easily checked that this is an equivalence relation. Hence we may define a *fiber bundle* as an equivalence class (under the above relation) of coordinate bundles.

Remark 4.8. It is worth noting at this point that an $\mathbf{R}^n$-bundle is a fiber bundle with fiber $\mathbf{R}^n$ and structure group $GL(n, \mathbf{R})$, an oriented $\mathbf{R}^n$-bundle is a fiber bundle with fiber $\mathbf{R}^n$ and structure group $GL_+(n, \mathbf{R})$ (matrices with positive determinant), and finally, a $\mathbf{C}^n$-bundle is a fiber bundle with fiber $\mathbf{C}^n$ and structure group $GL(n, \mathbf{C})$.

Definition 4.9. We say that the structure group G of a fiber bundle can be reduced to a subgroup $H \subseteq G$ if a local trivialization $\{(U_i, \phi_i)\}_{i\in I}$ can be chosen so that $g_{ij}(b) \in H$ for every $i, j \in I$ and $b \in U_i \cap U_j$. In this case, we may regard the bundle as a fiber bundle with structure group H.

Remark 4.10. The structure group of any $\mathbf{R}^n$-bundle over a paracompact space can be reduced to $O(n)$. Indeed, fixing an Euclidean metric on the vector bundle and using the Gram-Schmidt process, allows us to choose coordinate functions that send an orthonormal basis of $\mathbf{R}^n$ to an orthonormal basis of each fiber. Similarly, the structure group of any oriented $\mathbf{R}^n$-bundle over a paracompact space can be reduced to $SO(n)$. As all the spaces we consider in this work are paracompact, we will assume the latter reduction for the oriented vector bundles of this subsection.

As promised, the isomorphism class of fiber bundles over $\mathbb{S}^n$ with structure group G is determined by the homotopy class of a certain map. We now explain how this map is constructed.

Definition 4.11. Let ξ be a fiber bundle over $\mathbb{S}^n$ with group G. One can always take a trivialization of the bundle consisting of two coordinate neighborhoods $V_1, V_2 \subseteq \mathbb{S}^n$ that contain the equator $\mathbb{S}^{n-1} \subseteq \mathbb{S}^n$. Furthermore, one can choose a point $x_0 \in \mathbb{S}^{n-1}$ such that the coordinate transformation $g_{12} : V_1 \cap V_2 \to G$ sends x_0 to the zero element $e \in G$.[3] Restricting g_{12} to the equator $\mathbb{S}^{n-1}$ gives a function

$$T_\xi = g_{12|\mathbb{S}^{n-1}} : (\mathbb{S}^{n-1}, x_0) \to (G, e)$$

that will be called *characteristic map* of ξ.

Theorem 4.12. *(Classification of bundles over $\mathbb{S}^n$) Two fiber bundles ξ, η over $\mathbb{S}^n$ with common fiber F and path-connected group G are isomorphic if, and only if, their characteristic maps T_ξ and T_η are homotopic. Fiber bundles over $\mathbb{S}^n$ with group G are thus classified up to isomorphism by the homotopy group $\pi_{n-1}(G)$.*

A proof can be found in section 18 of [Ste51] or in chapter 1 of [Hat17].

For the vector bundles ξ_{hj} we have defined above, the equator $\mathbb{S}^3 \subseteq \mathbb{S}^4$ is identified under stereographic projection with the unit quaternions, also denoted $\mathbb{S}^3 \subseteq \mathbf{R}^4$. Using this, we obtain characteristic maps

$$f_{hj} := T_{\xi_{hj}} : (\mathbb{S}^3, 1) \to (SO(4), \mathrm{id})$$

given by $f_{hj}(u) \cdot v = u^h v u^j$ for any $u \in \mathbb{S}^3, v \in \mathbf{R}^4$.

We will need the following lemma.

Lemma 4.13. *Denote by η_f the $\mathbf{R}^m$-bundle over $\mathbb{S}^n$ with characteristic map $f : (\mathbb{S}^{n-1}, x_0) \to (SO(m), id)$ and assume that m is even. Then*

$$\eta_f \oplus \eta_g \cong \eta_{fg} \oplus \epsilon^m,$$

where ϵ^m is the trivial $\mathbf{R}^m$-bundle over $\mathbb{S}^n$ and fg is obtained by pointwise matrix multiplication, i.e. $(fg)(u) = f(u) \cdot g(u)$.

Proof. (from [Hat17]) The bundle $\xi_f \oplus \xi_g$ has characteristic map $f \oplus g : \mathbb{S}^{n-1} \to SO(2m)$ such that $(f \oplus g)(u)$ consists of $f(u)$ in the upper left block and of $g(u)$ in the lower right one. Since $SO(2m)$ is path-connected, we may consider the path $\gamma : [0,1] \to SO(2m)$ that starts with the identity matrix and ends with the matrix that acts by $(u_1, u_2) \mapsto (u_2, u_1)$ for any $u_1, u_2 \in \mathbf{R}^m$. Notice that the latter matrix lives in $SO(2m)$ because m is even. Then, the product $(f \oplus \mathrm{id})\gamma(t)(\mathrm{id} \oplus g)\gamma(t)$ gives a homotopy from $f \oplus g$ to $fg \oplus \mathrm{id}$, which is the characteristic map of $\eta_{fg} \oplus \epsilon^m$. □

Partial computation of $p_1(\xi_{hj})$

As a first step, let us show the following.

Lemma 4.14. *$p_1(\xi_{hj})$ is linear with respect to h and j.*

Proof. Notice that the pointwise matrix multiplication $f_{hj}(u) \cdot f_{h'j'}(u)$ equals $f_{h+h',j+j'}(u)$. By Lemma 4.13 and the properties of Pontrjagin classes, we have

$$p(\xi_{h+h',j+j'}) = p(\xi_{h+h',j+j'} \oplus \epsilon^4) = p(\xi_{hj} \oplus \xi_{h'j'}) = p(\xi_{hj})p(\xi_{h'j'}).$$

Keeping only elements of degree 1 (i.e. belonging to $H^4(\mathbb{S}^4)$), we obtain $p_1(\xi_{h+h',j+j'}) = p_1(\xi_{hj}) + p_1(\xi_{h'j'})$. □

[3] This is shown in section 18 of [Ste51].

Lemma 4.15. *$p_1(\xi_{hj}) = c(h-j)\iota$, where c is a suitable integer and ι is a generator of $H^4(\mathbb{S}^4)$.*

Proof. By the lemma above, we can already write $p_1(\xi_{hj}) = (ah+bj)\iota$. Notice that the assignments $(u,v) \mapsto (u,\bar{v})$, $(u',v') \mapsto (u',\overline{v'})$ determine an isomorphism $\xi_{hj} \to \xi_{-j-h}$. This is well-defined because

$$\overline{v'} = \frac{\overline{u^h v u^j}}{\|u\|^{h+j}} = \frac{\bar{u}^j \bar{v}\bar{u}^h}{\|u\|^{h+j}} = \frac{u^{-j}\bar{v}u^{-h}}{\|u\|^{-h-j}}.$$

Hence, $ah + bj = -aj - bh$ for every pair of integers h, j. Rearranging, we can write $(a+b)(h-j) = 0$ from which it is clear that $a = -b$. Setting $c = a$ finishes the proof. □

Determination of the signature $\sigma(B_k^8)$

The map $\rho_k : B_k^8 \to \mathbb{S}^4$ obtained by restricting $\pi_{hj} : E_{hj} \to \mathbb{S}^4$ to B_k^8 is a homotopy equivalence with homotopy inverse the zero-section. Hence, $\alpha = \rho_k^*(\iota)$ is a generator of $H^4(B_k^8)$ and so is $\beta = (i^*)^{-1}\alpha$ a generator of $H^4(B_k^8, M_k^7)$. We claim that β^2 is also a generator of $H^8(B_k^8, M_k^7)$. Indeed, it suffices to check that the cup product pairing

$$\begin{aligned} H^4(B_k^8, M_k^7) \times H^4(B_k^8, M_k^7) &\to \mathbf{Z} \\ (\beta_1, \beta_2) &\mapsto \langle \beta_1 \smile \beta_2, [B_k^8, M_k^7] \rangle \end{aligned}$$

is non-singular. This is a consequence of the fact that the sequence of isomorphisms

$$H^4(B_k^8, M_k^7) \overset{i^*}{\cong} H^4(B_k^8) \overset{h}{\cong} \mathrm{Hom}(H_4(B_k^8), \mathbf{Z}) \overset{D^*}{\cong} \mathrm{Hom}(H^4(B_k^8, M_k^7), \mathbf{Z})$$

takes a class $\beta_2 \in H^4(B_k^8, M_k^7)$ to the assignment $\beta_1 \mapsto \langle \beta_1 \smile \beta_2, [B_k^8, M_k^7] \rangle$. Here, the map h is the one in the universal coefficient theorem for cohomology 1.39 and D^* is the $\mathbf{Z}$-dual of the relative Poincaré duality isomorphism in Theorem 1.63.

We can thus choose the orientation of B_k^8 given by the fundamental class $[B_k^8, M_k^7] \in H_8(B_k^8, M_k^7)$ so that

$$\left\langle ((i^*)^{-1}\alpha)^2, [B_k^8, M_k^7] \right\rangle = +1. \tag{4.4}$$

In particular, this choice implies $\sigma(B_k^8) = +1$.

Partial computation of $p_1(B_k^8)$

We now fix integers h, j such that $h + j = 1$ and $h - j = k$.

Lemma 4.16. *We have an isomorphism of vector bundles*

$$\tau(B_k^8) \cong (\pi_{hj}^* \xi_{hj})_{|B_k^8} \oplus \rho_k^* \tau(\mathbb{S}^4)$$

Proof. After fixing a Riemannian metric on B_k^8, we may decompose $\tau(B_k^8)$ as the Whitney sum of $\tau_{D^4}(B_k^8)$ (the vectors tangent to the fiber D^4) and $\tau_{D^4}(B_k^8)^\perp$ (the vectors normal to the fiber D^4). Since $\rho_k : B_k \to \mathbb{S}^4$ is a submersion, by the regular value theorem, we have $\tau_{D^4}(B_k^8)^\perp \cong \rho_k^*\tau(\mathbb{S}^4)$. We can thus already write

$$\tau(B_k^8) \cong \tau_{D^4}(B_k^8) \oplus \rho_k^*\tau(\mathbb{S}^4). \tag{4.5}$$

We now consider the vector bundle ξ_{hj}. We claim that the bundle $\tau_{\mathbf{R}^4}(E_{hj})$ consisting of vectors of $\tau(E_{hj})$ that are tangent to the fiber $\mathbf{R}^4$ is isomorphic to $\pi_{hj}^*\xi_{hj}$. Indeed, since real vector spaces V are canonically identified with their tangent space T_vV for any $v \in V$, we may identify the fibers $F_e(\tau_{\mathbf{R}^4}(E_{hj})) = T_e(F_{\pi(e)}(\xi_{hj}))$ with the fibers $F_e(\pi_{hj}^*\xi_{hj}) = F_{\pi(e)}(\xi_{hj})$ yielding the desired isomorphism.

Finally, observe that $\tau_{D^4}(B_k^8) = \tau_{\mathbf{R}^4}(E_{hj})_{|B_k^8} \cong (\pi_{hj}^*\xi_{hj})_{|B_k^8}$, which, upon substituting in (4.5), finishes the proof. □

Now, by the Whitney product formula, we have

$$p_1(B_k^8) = p_1\left((\pi_{hj}^*\xi_{hj})_{|B_k^8}\right) + p_1\left(\rho_k^*\tau(\mathbb{S}^4)\right).$$

The first summand can be determined by naturality of the Pontrjagin classes under the sequence of bundle maps

$$(\pi_{hj}^*\xi_{hj})_{|B_k^8} \xrightarrow{i} \pi_{hj}^*\xi_{hj} \xrightarrow{\pi_{hj}} \xi_{hj},$$

where $i : B_k^8 \hookrightarrow E_{hj}$ is the inclusion. Since $\rho_k = \pi_{hj} \circ i$, we have

$$p_1\left((\pi_{hj}^*\xi_{hj})_{|B_k^8}\right) = \rho_k^* p_1(\xi_{hj}) = \rho_k^*(c(h-j)\iota) = ck\alpha.$$

For the second summand, taking elements of degree 1 in Theorem 3.61, we have

$$p_1(\mathbf{H}P^n) = (2n-2)u \tag{4.6}$$

Since $\mathbb{S}^4$ is diffeomorphic to the quaternionic projective line $\mathbf{H}P^1$, naturality implies $p_1\left(\rho_k^*\tau(\mathbb{S}^4)\right) = 0$.
Hence, $p_1(B_k^8) = ck\alpha$.

Determination of the constant c

We now restrict ourselves to the case $k = 1$. Consider the disk $D^8 = \{[u : v : 1] \mid \|u\|^2 + \|v\|^2 \le 1\} \subseteq \mathbf{H}P^2$. The assignments

$$(u, v) \mapsto [\bar{u} : 1 : \sqrt{1 + \|u\|^2}v]$$
$$(u', v') \mapsto [1 : u' : \sqrt{1 + \|u'\|^2}v']$$

define a diffeomorphism $B_1^8 \to \mathbf{H}P^2 \setminus D^8$.

Now, by equation (4.6), $p_1(\mathbf{H}P^2)$ equals twice a generator of $H^4(\mathbf{H}P^2)$. Since excision provides an isomorphism $H^4(\mathbf{H}P^2) \to H^4(\mathbf{H}P^2 \setminus D^8)$ induced by inclusion, we also have that $p_1(\mathbf{H}P^2 \setminus D^8)$ is twice a generator of $H^4(\mathbf{H}P^2 \setminus D^8)$. By the diffeomorphism $B_1^8 \cong \mathbf{H}P^2 \setminus D^8$, $p_1(B_1^8)$ is two times a generator of $H^4(B_1^8)$ as well. Finally, since $p_1(B_1^8) = c\alpha$, we must have $c = \pm 2$.

After all the previous results, we may now state and prove the theorem that was the main goal of this work.

Theorem 4.17. *$\lambda(M_k^7) \equiv k^2 - 1 \pmod 7$. Hence, whenever $k^2 \not\equiv 1 \pmod 7$, M_k^7 is an exotic 7-sphere.*

Proof. By the choice we made in equation (4.4), we have $q(B_k^8) = \langle ((i^*)^{-1}(\pm 2k\alpha))^2, [B_k^8, M_k^7] \rangle = 4k^2$. Substituting, we obtain

$$\lambda(M_k^7) \equiv 8k^2 - 1 \equiv k^2 - 1 \pmod 7.$$

$\square$

Bibliography

[GH81] Marvin J. Greenberg and John R. Harper. *Algebraic Topology: A First Course*. The Benjamin/Cummings Publishing Company, 1981.

[Hat01] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2001.

[Hat17] Allen Hatcher. *Vector Bundles and K-Theory*. 2017.

[Hir53] Friedrich Hirzebruch. Über die quaternionalen projektiven räume. pages 301–312, 1953.

[Hir78] Friedrich Hirzebruch. *Topological methods in algebraic topology*. Springer, 1978.

[Lee12] John M. Lee. *Introduction to Smooth Manifolds*. Springer, New York, 2012.

[May99] J. P. May. *A Concise Course in Algebraic Topology*. University of Chicago Press, 1999.

[Mil56] John W. Milnor. On manifolds homeomorphic to the 7-sphere. *Annals of Mathematics*, 64(2):399–405, 1956.

[Mil69] John W. Milnor. *Morse Theory*. Princeton University Press, 1969.

[Mil07] John W. Milnor. *Differential Topology*. American Mathematical Society, 2007.

[MS74] John W. Milnor and James D. Stasheff. *Characteristic classes*. Princeton University Press, 1974.

[Ste51] Norman Steenrod. *The Topology of Fibre Bundles*. Princeton University Press, 1951.

[Sto68] Robert E. Stong. *Notes on cobordism theory*. Princeton University Press, 1968.

[Tho54] René Thom. Quelques propriétés globales des variétés différentiables. *Commentarii Mathematici Helvetici*, (28):17–86, 1954.